Lecture 4: Mixing in the presence of sources and sinks

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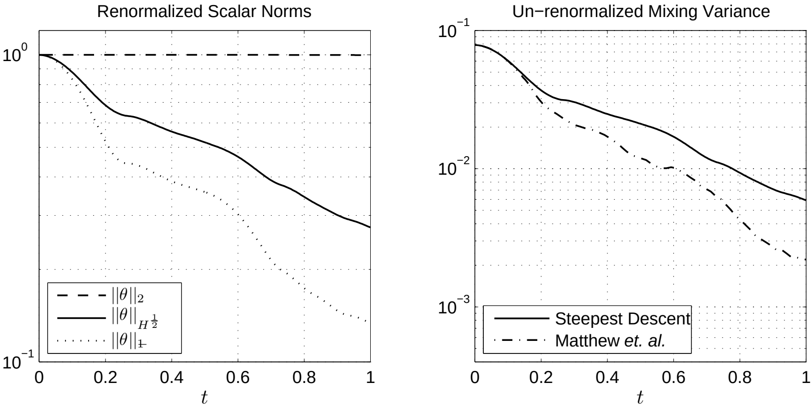
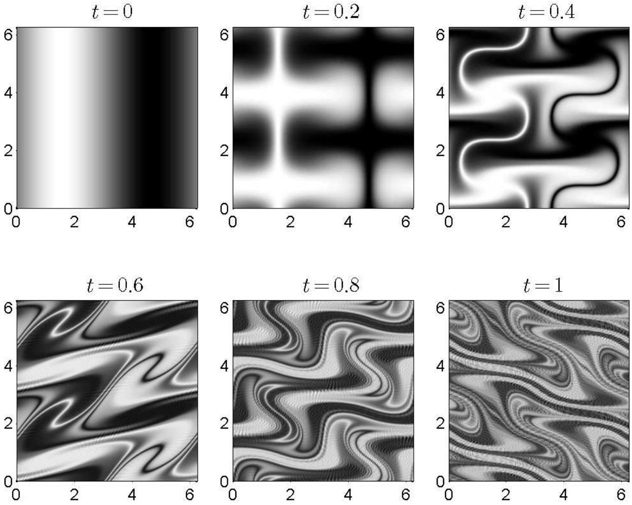
28 June 2010

Optimal control vs steepest descent



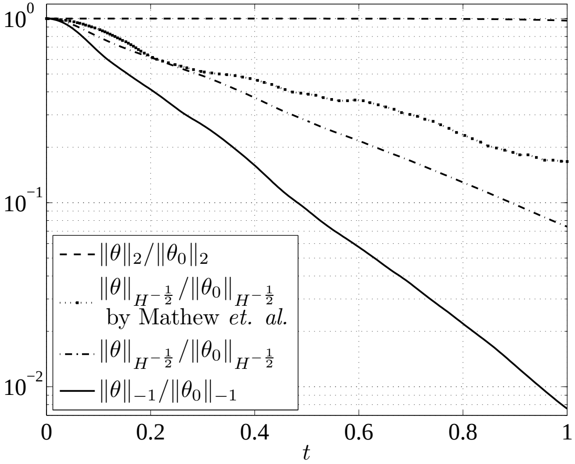
(from Lin, Thiffeault, Doering.)

Steepest descent of $\dot{H}^{-1}$



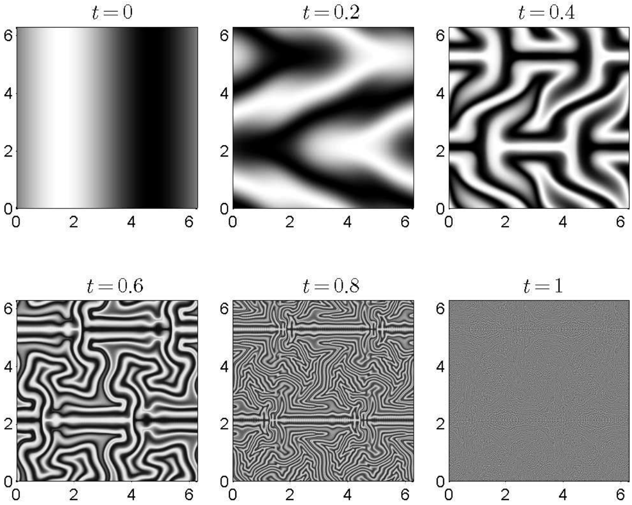
(from Lin, Thiffeault, Doering.)

Optimal control vs steepest descent: any flow



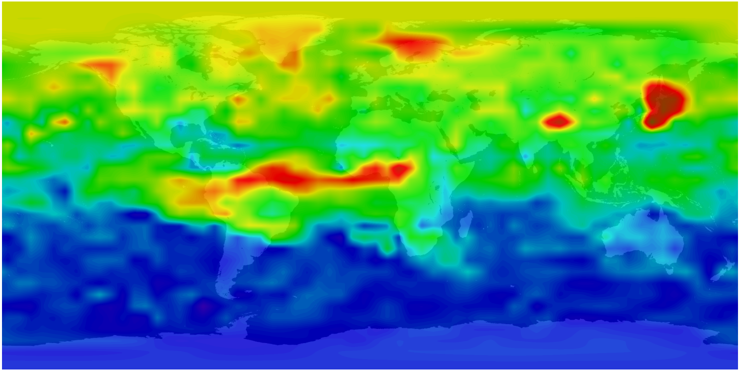
(from Lin, Thiffeault, Doering.)

Steepest descent of $\dot{H}^{-1}$: any flow



(from Lin, Thiffeault, Doering.)

Sources and sinks: CO in the atmosphere



Red corresponds to high levels of CO (450 parts per billion) and blue to low levels (50 ppb). Note the immense clouds due to grassland and forest fires in Africa and South America. (Photo NASA/NCAR/CSA.)

Matlab code: Minimize norm with fmincon

```
function [psi,Effq] = velopt(psi0,src,kappa,q,L,scalefac)

% Problem parameters for Matlab's optimizer fmincon.
psi0 = psi0(:); problem.x0 = psi0(2:end);
problem.objective = @(x) normHq2(x,src,kappa,q,L,scalefac);
problem.nonlcon = @(x) nonlcon(x,src,kappa,q,L,scalefac);
problem.solver = 'fmincon';
problem.options = optimset('Display','iter','TolFun',1e-10,...
   'GradObj','on','GradConstr','on',...
   'algorithm','interior-point');

[psi,Hq2] = fmincon(problem);

% Mixing efficiency: call normHq2 with no flow to get pure-conduction solution.
Effq = sqrt(normHq2(zeros(size(psi)),src,kappa,q,L,scalefac) / Hq2);

psi = reshape([0;psi],size(src)); % Convert psi back into a square grid
```

Matlab code: Right-hand side function

```
function [varargout] = normHq2(psi,src,kappa,q,L,scalefac)

N = size(src,1); src = src(:);

% 2D Differentiation matrices and negative-Laplacian
[Dx,Dy,Dxx,Dyy] = Diffmat2(N,L); mlap = -(Dxx+Dyy);
if q ~= 0 && q ~= -1, error('This code only supports q = 0 or -1.'); end

psi = [0;psi]; ux = Dy*psi; uy = -Dx*psi;
ugradop = diag(sparse(ux))*Dx + diag(sparse(uy))*Dy;

if q == 0
  Aop2 = (-ugradop + kappa*mlap);
elseif q == -1
  Aop2 = mlap*(-ugradop + kappa*mlap);
end
Aop1 = (ugradop + kappa*mlap)*Aop2;
% Solve for chi, dropping corner point to fix normalisation.
chi = [0; Aop1(2:end,2:end) \ src(2:end)];
theta = Aop2*chi;

% The squared H^q norm of theta.
varargout{1} = L^2*sum(theta.^2)/N^2 * scalefac;

if nargout > 1
  % Gradient of squared-norm Hq2.
  gradHq2 = 2*((Dx*theta).*(Dy*chi) - (Dy*theta).*(Dx*chi));
  varargout{2} = gradHq2(2:end) / N^2 * scalefac;
end
```

Matlab code: Constraints

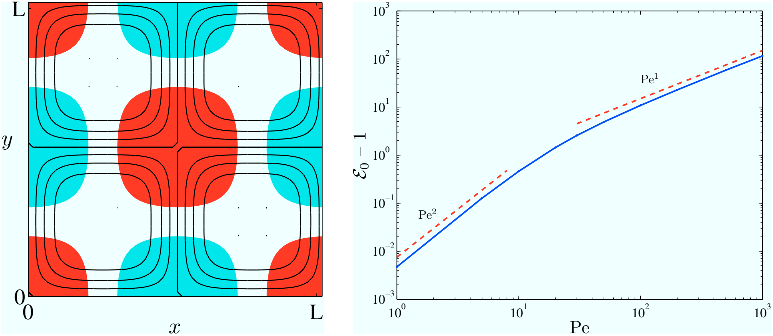
```
function [c,ceq,gc,gceq] = nonlcon(psi,src,kappa,q,L,scalefac)

psi = [0;psi]; N = size(src,1);
c = []; gc = [];

[Dx,Dy,Dxx,Dyy] = Diffmat2(N,L); % 2D Differentiation matrices
U2 = L^2*(sum((Dx*psi).^2 + (Dy*psi).^2)/N^2);
ceq(1) = (U2-1) * scalefac;

if nargout > 2
  % Gradient of constraints
  mlappsi = -(Dxx+Dyy)*psi;
  gceq(:,1) = 2*mlappsi(2:end) / N^2 * scalefac;
end
```

Optimal stirring flow



Left: Optimal stirring velocity field (streamlines) for the source $\sin x \sin y$, for $\mathrm{Pe} = 10$. Right: Dependence on Péclet number of the optimal mixing efficiency $\mathcal{E}_0$. For small Pe the optimal streamfunction $\to (\sqrt{2}\pi)^{-1} \cos x \cos y$.

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