# GFD Lectures: Swimming & Swirling

2010/06/28

## Lecture 4: Mixing in the Presence of Sources and Sinks

### Part 1: Norms

$$\partial_t \theta + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \,, \qquad \nabla \cdot \underline{u} = 0 \qquad (AD)$$

in $\Omega$ a
- bounded domain with zero-flux conditions
- OR
- periodic domain

Assume $\int_\Omega \theta \, d\Omega = 0$. Let $\|\theta\|_2^2 = \int_\Omega \theta^2 d\Omega$ $L^2$-norm Also VARIANCE

Recall: $\frac{d}{dt} \|\theta\|_2^2 = -2\kappa \|\nabla \theta\|_2^2$ Equation of variance decay.

Variance ($L^2$-norm) would seem a good measure of mixing. But it requires knowledge of small-scales in $\theta$, which we may not care about. Wouldn't it be better to blindly solve:

$$(A) \qquad \frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = 0 \quad ?$$

since often we don't care how something is homogenized

But then: $\frac{d}{dt} \|\theta\|_2^2 = 0$, so can't use variance.

The advection equation (A) takes us closer to the ergodic theory sense of mixing.

In ergodic theory, we think of an operator $S^t : \Omega \to \Omega$, which is obtained from the solution of (A). $S^t$ "moves forward" a patch of dye $\theta_0(\underline{x})$ to $\theta(\underline{x}, t)$.

For a patch A, $Vol(A)$ (or $Area(A)$) is the <u>Lebesgue measure</u> of A

$\theta(\underline{x}, t)$ only 0 or 1, say

Because of incompressibility, $S^t A$ has the same volume as A. $S^t$ is <u>measure-preserving</u>.

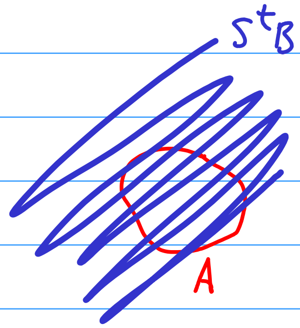
Now for the definition of <u>MIXING</u> (in the sense of ergodic theory):

$$\boxed{\lim_{t\to\infty} Vol(A \cap S^t(B)) = Vol(A)\,Vol(B)}$$

for all patches A, B in $\Omega$.

usual def'n has $-t$, but S invertible here so it's the same.

What does this mean? Imagine B has been mixed:




Recall $S^t B$ has the same volume as B. So if it "fills the space" $\Omega$ as best as it can, its intersection with A can only be as large as $Vol(B)$, but also $Vol(A)$. This is true for any A, B, so every blob must spread everywhere.

This is actually called STRONG MIXING. It implies ergodicity, but not the other way around.

Notice that this follows our intuition for what "good mixing" is, but no diffusion is needed.

In fact, the arbitrary "reference patch" $A$ is a bit like a function that we project on. This suggests another def'n, which is more "analytic":

Weak convergence:

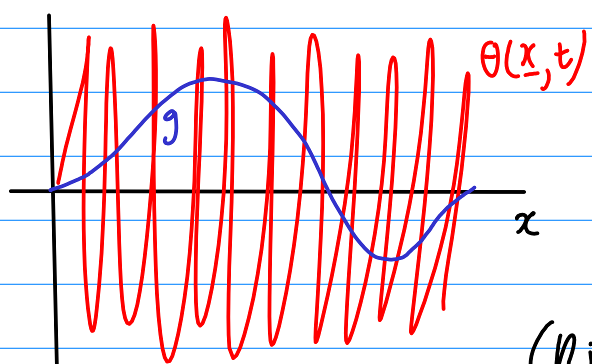
$$\lim_{t \to \infty} \langle \theta(\underline{x}, t), g \rangle = 0$$

($\theta$ converges to zero weakly)

for all functions $g(\underline{x})$ in $L^2(\Omega)$

Here: $\langle f, g \rangle = \int_\Omega f(\underline{x})\, g(\underline{x})\, d\Omega$, and a function $f$ is in $L^2(\Omega)$ if $\int_\Omega |f|^2 d\Omega < \infty$ (for example, $\delta$-functions are not)

Weak convergence is equivalent to mixing. Why?




$\theta$ is no vanishing, but it is getting wigglier, so

$$\int \theta g \, dx \to 0$$

(Riemann-Lebesgue lemma)

But neither the def'n of mixing and weak convergence are that useful in practice: hard to compute something over all functions $g(\underline{x})$!

But there is a simpler way: Mathew, Mezic, & Petzold introduced the mix-norm, which is basically a negative Sobolev norm:

The "dot" is because we assume $\int \theta = 0$

$$\|\theta\|_{\dot{H}^q} := \|\nabla^q \theta\|_2$$

Sobolev norm for $\dot{H}^q(\Omega)$ ($q=0$ is $L^2$ norm)

We can interpret this norm for $q<0$ as well! This is easiest on a periodic domain:

$$\boxed{\|\theta\|_{\dot{H}^q}^2 = \sum_{\underline{k}}{}' |\underline{k}|^{2q} |\hat{\theta}_{\underline{k}}|^2}$$

Note $\hat{\theta}_{\underline{0}} = 0$ (mean)

For $q<0$, $\|\theta\|_{\dot{H}^q}$ smooths $\theta$ before taking the $L^2$ norm.

Theorem (Mathew-Mezic-Petzold, Doering-Lin-T)

$$\lim_{t\to\infty} \|\theta\|_{\dot{H}^q} = 0, \quad q<0 \iff \theta \text{ converges weakly to } 0.$$

(proof is short, but a bit technical.)

Upshot: we can track any of these norms to determine if a system is mixing. $q$ controls how much smoothing is imposed.

This makes optimization easier, for instance.

Time-evolution of $\dot{H}^q$-norms: (w/o diffusion)

$$\nabla^{-1} \sim \frac{-i\underline{k}}{|\underline{k}|^2}$$

$$\frac{d}{dt}\|\theta\|_{\dot{H}^{-1}}^2 = \langle \nabla^{-1}\theta \cdot \nabla u \cdot \nabla^{-1}\theta \rangle$$

$$\frac{d}{dt}\|\theta\|_{\dot{H}^{1}}^2 = -\langle \nabla\theta \cdot \nabla u \cdot \nabla\theta \rangle$$

NOT conserved even in the absence of diffusion.

(other norms are uglier)

Mathew, Mezic, Grivopoulos, Vaidya, Petzold: use optimal control to optimize decay of $\|\theta\|^2_{\dot{H}^{-1/2}}$ (nonlocal in time)

Lin, Doering, T: maximize instantaneous decay rate of $\|\theta\|^2_{\dot{H}^{-1}}$

(local in time, easier, almost as good)

SLIDES pages 2-5

# PART 2: SOURCES AND SINKS

$$\partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta + \underbrace{s(\underline{x},t)}_{\substack{\text{sources}/\text{sinks}\\ >0 \quad <0}} \qquad \text{(ADs)}$$

$$(\nabla\cdot\underline{u}=0) \qquad \text{(SLIDE p. 6)}$$

Assume: $\int_\Omega s(\underline{x},t)\,d\Omega = 0$ (otherwise subtract the mean)

More convenient to think of hot/cold
sources sinks

For simplicity, restrict to time-independent $s(\underline{x})$.
Then system achieves a steady-state: (unlike decaying problem)

$$\underline{u}\cdot\nabla\theta = \kappa\nabla^2\theta + s \qquad \text{let } \mathcal{L} \equiv \underline{u}\cdot\nabla - \kappa\nabla^2$$

$$\hookrightarrow \quad \mathcal{L}\theta = s \quad \text{or} \quad s = \underbrace{\mathcal{L}^{-1}}_{\text{integral operator}}\theta$$

Note that $\kappa \neq 0$ is needed to reach steady-state.

So, assuming the system has reached a steady-state, how do we measure the "quality of mixing"?

Can look at norms $\|\theta\|_{\dot{H}^q}$ ($q=0$ is standard deviation)

But what do we compare to?

One possibility: $\dfrac{\|\theta\|_{\dot{H}^q}}{\|s\|_{\dot{H}^q}}$ Pretty good, but has units of inverse time.

Prefer <u>mixing enhancement factors:</u>

$$\boxed{\mathcal{E}_q = \frac{\|\tilde{\theta}\|_{\dot{H}^q}}{\|\theta\|_{\dot{H}^q}}}$$

$$\tilde{\mathcal{L}} = -\kappa\nabla^2$$

$$\tilde{\mathcal{L}}\tilde{\theta} = s$$

$\tilde{\theta}$ is the solution in the absence of stirring. (purely diffusive)

Since $\|\theta\|_{\dot{H}^q}$ is usually <u>decreased</u> by stirring, $\mathcal{E}_q$ measures the enhancement over the pure-diffusion state.

Several properties given in Doering & T, Shaw, Doering, & T.

For instance, can we have $\mathcal{E}_q < 1$, i.e., can stirring ever be worse than not stirring?

Consider $\mathcal{E}_1 = \dfrac{\|\nabla\tilde{\theta}\|_2}{\|\nabla\theta\|_2}$.

$$\tilde{\theta} = \tilde{\mathcal{L}}^{-1}s = (-\kappa\nabla^2)^{-1}s = -\kappa^{-1}\nabla^{-2}s \Rightarrow \nabla\tilde{\theta} = -\kappa^{-1}\nabla^{-1}s.$$

Also: $\mathcal{L}\theta = s \Rightarrow \langle\theta\,\mathcal{L}\theta\rangle = \langle s\theta\rangle \qquad \langle\cdot\rangle = \int_\Omega \cdot\, d\Omega$

$$\underbrace{\cancel{\langle\theta\, u\cdot\nabla\theta\rangle}}_{=\langle\nabla\cdot(u\theta^2/2)\rangle=0} - \kappa\langle\theta\nabla^2\theta\rangle = \langle s\theta\rangle$$

$$\kappa\langle|\nabla\theta|^2\rangle = \langle\theta s\rangle = \langle\theta\,\nabla\cdot\nabla^{-1}s\rangle$$

$$\kappa\|\theta\|_{\dot{H}^1}^2 = -\langle\nabla\theta\cdot\nabla^{-1}s\rangle = \kappa\langle\nabla\theta\cdot\nabla\tilde{\theta}\rangle$$

$$\|\theta\|_{\dot{H}^1}^2 = \langle\nabla\theta\cdot\nabla\tilde{\theta}\rangle \leq \underbrace{\|\nabla\theta\|_2}_{\|\theta\|_{\dot{H}^1}}\underbrace{\|\nabla\tilde{\theta}\|_2}_{\|\tilde{\theta}\|_{\dot{H}^1}}$$

Cauchy-Schwartz inequality

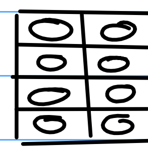
$$\therefore \ \|\theta\|_{\dot H^1} \leqslant \|\tilde\theta\|_{\dot H^1} \iff \boxed{\mathcal{E}_1 \geqslant 1}$$

This is somewhat counter-intuitive: gradients are usually <u>increased</u> by stirring! However, here we're talking about gradients in a steady-state, affected by diffusion.

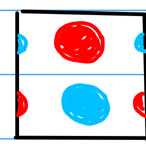
What about the other ones, $\mathcal{E}_q$, $q \neq 1$? Do we have $\mathcal{E}_q \geqslant 1$?

We tried and failed to prove this, because it isn't true. Following a challenge by Charlie Doering, Jeff Weiss came up with something like:

$$\underline{u} = (2\sin x \cos 2y, \ -\cos x \sin 2y)$$



$$s = \left(\cos x - \tfrac{1}{2}\right) \sin y$$



(Péclet = 4)

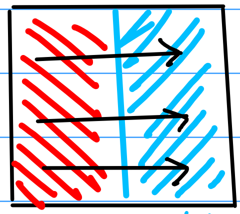
This manages to "concentrate" the source sink distribution more that under pure diffusion, and

$$\mathcal{E}_0 \simeq .978, \qquad \mathcal{E}_{-1} \simeq .945$$

Slightly less than 1! Not a dramatic effect, but it's there!
(more later)

OPTIMIZATION: What kinds of flow give the largest $\mathcal{E}_q$, given source/sink distribution $s(\underline{x})$? (FIXED ENERGY)

Surprising example: $s(\underline{x}) = \sin x$ (periodic B.C.)




optimal: $\underline{u} = U\hat{x}$ Constant flow!

(see Shaw-T-Doering, Plasting-Young)

This example demonstrates that with body sources the best stirring has more to do with transport than with creation of small scales.

SOLVE NUMERICALLY for more complicated sources. (SLIDE p.7, Matlab)