GFD Lectures: Swimming & Swirling 2010/06/24

Lecture 3: Local Stretching Theories

Antonsen et al. '96
Balkovsky & Fouxon '99

(AD) $\partial_t \theta + \underline{u}\cdot\nabla\theta = \kappa \nabla^2 \theta.$

For this lecture, think of θ as a "patch"

Last time we examined $\underline{u} = (\lambda x, -\lambda y)$. Let's try something more general:

$$\underline{u} = \underbrace{\underline{U}}_{\text{const}} + \underline{x}\cdot \underbrace{A}_{\text{const}}, \qquad \nabla\cdot\underline{u} = \operatorname{trace} A = 0.$$

Let $\langle f\rangle = \int_\Omega f\, dV \qquad (\Omega = \mathbb{R}^2 \text{ or } \mathbb{R}^3)$

Solve (AD) using moments: $\boxed{c_i = \dfrac{\langle x_i\theta\rangle}{\langle\theta\rangle}}$ $\qquad \left(\partial_t\langle\theta\rangle = 0\right)$

(AD) $\rightarrow$ $\partial_t\langle x_i\theta\rangle + \langle x_i \nabla\cdot((\underline{U}+\underline{x}\cdot A)\theta)\rangle = \kappa\langle x_i\nabla^2\theta\rangle$

$$\partial_t\langle x_i\theta\rangle - \langle (U_j + x_\ell A_{\ell j})\theta \cdot \underbrace{\partial_j x_i}_{\delta_{ji}}\rangle = \kappa\ (0)$$

$$\langle\theta\rangle\partial_t c_i - U_i\langle\theta\rangle - A_{\ell i}\langle\theta\rangle c_\ell = 0$$

$$\boxed{\partial_t \underline{c} = \underline{U} + \underline{c}\cdot A}$$

Motion of center of mass

Next moments: $\boxed{m_{ij} = \dfrac{\langle x_i x_j\theta\rangle}{\langle\theta\rangle} - c_i c_j}$

Again, multiply (AD) by $x_i x_j$ and $\langle\cdot\rangle$.

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = \langle x_i x_j \partial_k ((U_k + x_l A_{lk})\theta)\rangle$$

$$= -\langle (U_k + x_l A_{lk})(\delta_{ik} x_j + x_i \delta_{jk})\theta\rangle$$

$$= -U_i c_j \langle\theta\rangle - U_j c_i \langle\theta\rangle - A_{li} \underbrace{\langle x_l x_j \theta\rangle}_{\langle\theta\rangle(m_{lj} + c_l c_j)} - A_{lj} \underbrace{\langle x_l x_i \theta\rangle}_{\langle\theta\rangle(m_{li} + c_l c_i)}$$

$$\partial_t (c_i c_j) = c_i \partial_t c_j + c_j \partial_t c_i$$
$$= c_i (U_j + A_{lj} c_l) + c_j (U_i + A_{li} c_l)$$

$$\langle x_i x_j \nabla\cdot(\underline{u}\theta)\rangle = -\left(\partial_t (c_i c_j) + A_{li} m_{lj} + A_{lj} m_{li}\right)\langle\theta\rangle$$

That's the hard part! Next:

$$\langle x_i x_j \nabla^2\theta\rangle = \langle\theta\, \nabla^2 (x_i x_j)\rangle = 2\langle\theta\rangle \delta_{ij}$$

So finally:

$$\partial_t m_{ij} = A_{li} m_{lj} + A_{lj} m_{il} + 2\kappa \delta_{ij}$$

Let $(M)_{ij} = m_{ij}$ (symmetric matrix)

$$\boxed{\partial_t M = M\cdot A + A^T\cdot M + 2\kappa I}$$

Moment of inertia equation. "spread" of patch

Time to solve these equations!

$$\underline{c}(t) = \underline{c}(0)\cdot e^{At} + \underline{U}\cdot\int_0^t e^{A(t-\tau)}\, d\tau$$

$$M(t) = e^{A^T t} \cdot M(0) \cdot e^{At} + 2\kappa \int_0^t \underbrace{e^{A^T(t-\tau)} \cdot e^{A(t-\tau)}}_{\text{Can't write as } e^{(A+A^T)(t-\tau)}} d\tau$$

unless $[A, A^T] = 0$ Normal matrix

Let $M = RDR^T$, R orthogonal, D diagonal

$$\dot{M} = \dot{R}DR^T + RD\dot{R}^T + R\dot{D}R^T = RDR^TA + A^TRDR^T + 2\kappa I$$

$$R^T\dot{R}D + D\dot{R}^TR + \dot{D} = D\underbrace{R^TAR}_{\tilde{A}} + \underbrace{R^TA^TR}_{\tilde{A}^T}D + 2\kappa I$$

Now: $\frac{d}{dt}(R^TR) = \dot{R}^TR + R^T\dot{R} = \frac{d}{dt}(I) = 0$, so $(R^T\dot{R})^T = \dot{R}^TR = -R^T\dot{R}$

$$\Rightarrow R^T\dot{R} \text{ is antisymmetric}$$

$$[R^T\dot{R}D]_{ii} = (R^T\dot{R})_{ik} D_{ki} = (R^T\dot{R})_{ii} D_{ii} = 0$$

(no sum)

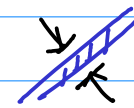
$$\dot{D}_{ii} = D_{il}\tilde{A}_{li} + \tilde{A}_{li} D_{li} + 2\kappa$$

$$\dot{D}_{ii} = 2\tilde{A}_{ii} D_{ii} + 2\kappa$$

Write $D_{ii} = e^{2\rho_i}$, with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_d$.

$$\dot{D}_{ii} = 2e^{2\rho_i}\dot{\rho}_i, \qquad \boxed{\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}}$$

Great equation: • $\tilde{A} = R^T A R \rightarrow$ rotated velocity gradient matrix.

• $e^{-2\rho_i} \rightarrow$ negligible unless $\rho_i < 0$ 

compression

Moral: the directions of contraction or compression play an important role.

Now we need an equation for R: off-diagonal terms.

$$[R^T \dot{R} D]_{ij} = (R^T \dot{R})_{il} D_{lj} = (R^T \dot{R})_{ij} D_{jj} \quad , \; i \neq j$$

$$[D \dot{R}^T R]_{ij} = D_{il} (\dot{R}^T R)_{lj} = D_{ii} (\dot{R}^T R)_{ij} = -(R^T \dot{R})_{ij} D_{ii}$$

(no sum over i, j)

$$(D_{jj} - D_{ii})(R^T \dot{R})_{ij} = D_{ii} \tilde{A}_{ij} + \tilde{A}_{ji} D_{jj}$$

$$(R^T \dot{R})_{ij} = \Omega_{ij} \iff \boxed{\dot{R} = R\Omega}$$

$$\boxed{\Omega_{ij} = \frac{e^{2\rho_i} \tilde{A}_{ij} + e^{2\rho_j} \tilde{A}_{ji}}{e^{2\rho_j} - e^{2\rho_i}}}$$

Not completely obvious what this means...

$(= 0 \text{ for } i = j)$

Almost always true for long time, esp. in 2D, 3D with $\rho_1 + \rho_2 (+\rho_3) = 0$.
Usually a symmetry can break this, or fails locally.

Assume we have separation between the eigenvalues: $e^{2\rho_i} \gg e^{2\rho_j}, \; i < j$

$$\Omega_{ij} \simeq \frac{e^{2\rho_i} \tilde{A}_{ij} + \cancel{e^{2\rho_j} \tilde{A}_{ji}}}{\cancel{e^{2\rho_j}} - e^{2\rho_i}} = -\tilde{A}_{ij} \quad , \; i < j$$

$$\boxed{\Omega_{ij} \simeq \begin{cases} -\tilde{A}_{ij}, & i<j \\ \tilde{A}_{ji}, & i>j \end{cases}}$$

(large t)

Independent of eigenvalues!

Can solve: $\dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i}$ since $\tilde{A}$ indep. of ρ_i

$$\rho_i(t) = \rho_{i0} + \mathcal{A}_i(t) + \frac{1}{2}\log\left[1 + \underset{\substack{\uparrow \\ \text{diffusion}}}{2\kappa}\, e^{-2\rho_{0i}} \int_0^t \exp\left(-2\mathcal{A}_i(t')\right) dt'\right]$$

where

$$\mathcal{A}_i = \int_0^t \tilde{A}_{ii}(t')\,dt'$$

When diffusion negligible: $\rho_i(t) = \rho_{i0} + \int_0^t \tilde{A}_{ii}(t')\,dt'$

In fact, solving the equations for ρ_i, R, $\underline{\underline{\kappa=0}}$, is not a bad way of computing Lyapunov exponents:

$$\boxed{\lambda_i = \lim_{t\to\infty} \frac{1}{t}\rho_i(t)} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$$

(Some numerical issues regarding orthogonality of R.)

Convergence: famous Oseledec Multiplicative ergodic theorem

Now comes the stochastic part: could have formulated things in terms of an SDE. But we take a shortcut:

$$\rho_i(t) = \rho_{0i} + \sum_t \tilde{A}_{ii} \quad \longleftarrow \text{sum of uncorrelated random numbers (more later)}$$

What is PDF of $p_i(t)$?

Recall: if x_i are i.i.d. and $X = \sum_{i=1}^{N} x_i$ $\qquad \overline{x_i} = \xi$, $\overline{x_i^2} - \overline{x_i}^2 = \sigma^2$

What is PDF of X? CENTRAL LIMIT THEOREM

mean of X

$$P(X,N) \sim \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left(-\frac{(X - N\xi)^2}{2N\sigma^2}\right)$$

Valid for: (i) $N \gg 1$; (ii) $X - N\xi < \sqrt{N}\sigma$

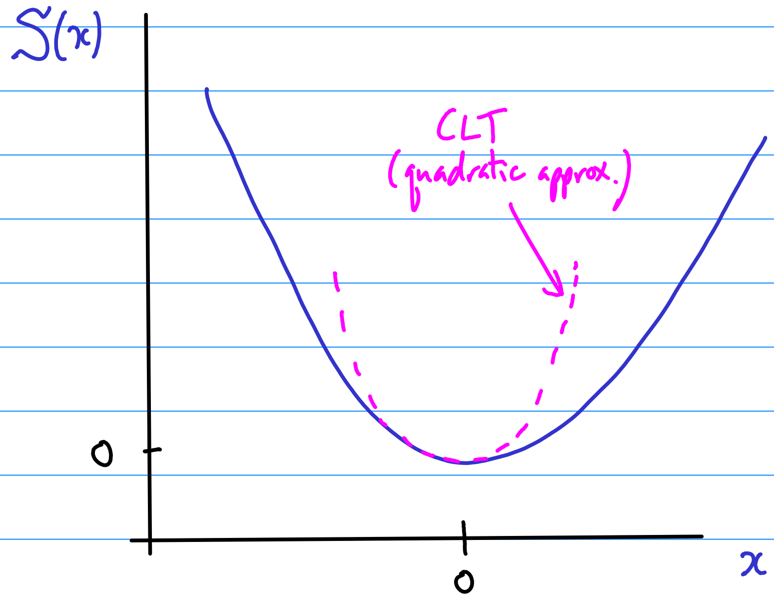
This second restriction is less commonly stated: it tells us that the CLT is not valid in the tails. The CLT tends to vastly underestimate the probability of rare events, or black swans as is trendy to call them these days. These tails matter for mixing.

More generally,

$$\boxed{P(X,N) \sim \exp\left(-N S\left(\frac{X-N\xi}{N}\right)\right)}$$

Large deviation form

$S(x)$ is a convex function with $S(0) = S'(0) = 0$.



$$S(x) = \overbrace{S(0)}^{0} + \overbrace{S'(0)}^{0}x + \frac{1}{2}S''(0)x^2 + \dots$$

$$S\left(\frac{X-N\xi}{N}\right) = \frac{1}{2}S''(0)\frac{(X-N\xi)^2}{N^2} + \dots$$

$$\exp\left(-NS\left(\frac{X}{N}-\xi\right)\right) \sim \exp\left(-S''(0)\frac{(X-N\xi)^2}{2N}\right)$$

Compare to CLT: $\boxed{S''(0) = \frac{1}{\sigma^2}}$

Can also express in terms of mean: $x = \frac{X}{N}$

$$P(x, N) \sim \exp\left(-NS(x-\xi)\right)$$

Example: Binomial distribution for x_i (−1 or 1, mean 0)

$$p(x_i) = \frac{1}{2}\delta(x_i+1) + \frac{1}{2}\delta(x_i+1)$$

$$e^{-s(k)} = \int p(\xi)\, e^{-ik\xi}\, d\xi \qquad \text{characteristic function}$$

$$= \frac{1}{2}\left(e^{ik} + e^{-ik}\right) = \cos k$$

For the mean $x = \frac{1}{N}\sum x_i$:

$$P(x,N) = \int p(x_1)\cdots p(x_N)\,\delta\left(\frac{x_1+\cdots+x_N}{N} - x\right) dx_1\cdots dx_N$$

$$e^{-S(k)} = \int P(x,N)\,e^{-ikx}\,dx$$

$$= \int p(x_1)\cdots p(x_N)\,e^{-ik(x_1+\cdots+x_N)/N}\,dx_1\cdots dx_N$$

$$= \prod_{i=1}^{N}\int p(x_i)\,e^{-ikx_i/N}\,dx_i = \left(\int p(\xi)\,e^{-ik\xi/N}\,d\xi\right)^N$$

$$= \left(e^{-s(k/N)}\right)^N = \cos^N(k/N)$$

Inverse Fourier

$$P(x,N) = \frac{1}{2\pi}\int e^{-S(k)}\,e^{ikx}\,dk = \frac{1}{2\pi}\int \cos^N\left(\frac{k}{N}\right)e^{ikx}\,dk$$

$$= \frac{N}{2\pi}\int \cos^N K\; e^{iNKx}\,dK\,, \quad K = k/N.$$

$$= \frac{N}{2\pi}\int e^{N(\log\cos K + iKx)}\,dK$$

For N large, look for saddle (stationary) point:

$$\frac{d}{dK}\underbrace{(\log\cos K + iKx)}_{H(K,x)} = -\tan K + ix = 0 \text{ when } K = K_{sp}.$$

$$\tan K_{sp} = -ix$$

$$H(k,x) = H(k_{sp},x) + \underbrace{H'(k_{sp},x)}_{0}(k-k_{sp}) + \frac{1}{2}H''(k_{sp},x)(k-k_{sp})^2 + \cdots$$

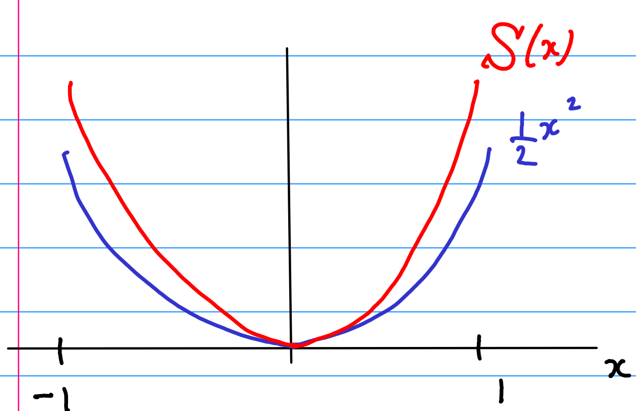
With this approximation the inverse transform is a Gaussian integral.
Get finally (skip some steps ... see Aosta lecture notes)

$$P(x,N) = \sqrt{\frac{N S''(0)}{2\pi}}\, e^{-NS(x)}, \quad \text{with}$$

$$\boxed{S(x) = -\frac{1}{2}(x+1)\log\left(\frac{1-x}{x+1}\right) + \log(1-x)} \qquad -1 \le x \le 1$$

Note $S(0) = 0$, $S'(x) = -\frac{1}{2}\log\left(\frac{1-x}{x+1}\right)$, so $S'(0) = 0$

$$S''(x) = \frac{1}{1-x^2}, \quad \text{so} \quad S''(0) = 1$$



$S(x)$ is called the
rate function
Cramer function
entropy function

For this case the Gaussian form overestimates the probability in the tails (not typical)

More refs:
Falkovich et al. 2001
Zeldovich et al. 1984

What this has to do with mixing?

For $\kappa = 0$, we argued that if A_{ii} is a random var., then ρ_i are distributed according to large deviation form (for large t).

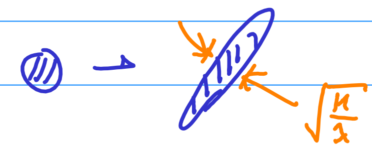
$$P(\rho_1, \rho_2, t) \sim \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) \underbrace{\theta(\rho_1)}_{\substack{\text{ordering} \\ \rho_1 \geq \rho_2}} \underbrace{\delta(\rho_1 + \rho_2)}_{\text{incompressibility}}$$

in 2D ($d = 2$). (return 3D later)

$\theta(x) =$ step function

$$\lambda_1 = \lim_{t \to \infty} \rho_1 / t = \text{Lyapunov exp.} > 0 \text{ (for chaotic flows)}$$

What happens with diffusion? Recall "filament":



The contracting direction "stabilizes" near the Batchelor width $\sqrt{\frac{\kappa}{\lambda_1}}$.

or "freezes"

Shraiman & Siggia 1994
Chertkov et al. 1997
Balkovsky & Fouxon 1999

$$P(\rho_1, \rho_2, t) \sim \exp\left(-t\, S\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) \underbrace{P_{stab}(\rho_2)}_{\text{stationary distribution.}}$$

If we assume, say, an initial Gaussian "patch" of passive scalar, then the concentration at a point scales as

$$\underbrace{\theta(\underline{x}, t)}_{\text{indep. of } \underline{x}} \sim \frac{\text{total concentration}}{\text{volume}} \sim (\det M)^{-1/2}$$

$$= \exp\left(-\sum \rho_i\right)$$

Expected value:

$$\langle \theta^\alpha \rangle(t) \sim \int e^{-\alpha \sum \rho_i} \exp\left(-tS\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) P_{stab}(\rho_2)\, d\rho_1\, d\rho_2$$

Non-exponential function of t (neglect)

$$\sim \int e^{-\alpha \rho_1} \exp\left(-tS\left(\frac{\rho_1 - \lambda_1 t}{t}\right)\right) d\rho_1$$

← Do the ρ_2 integral

Use $h_i = \rho_i / t$ as variable:

expected value, not integral

$$\langle \theta^\alpha \rangle(t) \sim \int e^{-\alpha h_1 t}\, e^{-tS(h_1 - \lambda_1)}\, dh_1$$

$h_1 \to h$

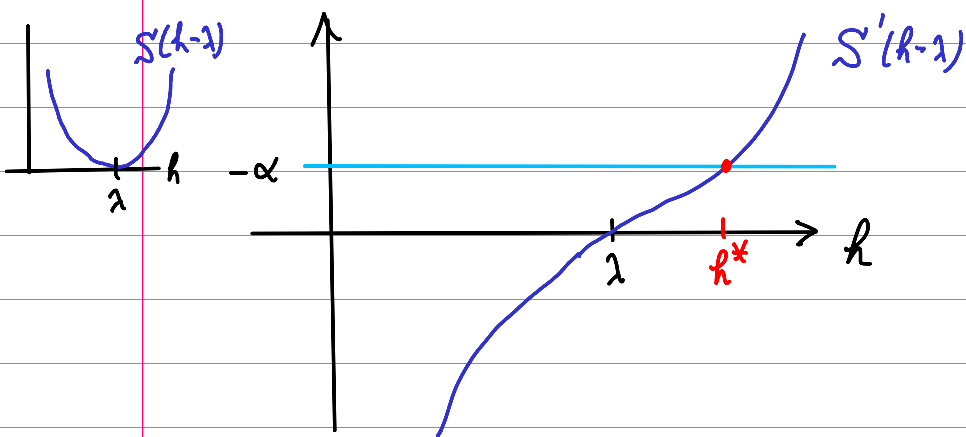
$\lambda_1 \to \lambda$

$$\boxed{\langle \theta^\alpha \rangle(t) \sim \int e^{-t(\alpha h + S(h - \lambda))}\, dh}$$

Let $H(h) = \alpha h + S(h-\lambda)$.

For large time, the integral is dominated by saddle point h^*:

$$H'(h^*) = 0 = \alpha + S'(h^*-\lambda)$$



Because of convexity of S, h^* is unique.

We then have $H(h) = H(h^*) + \frac{1}{2}H''(h^*)(h-h^*)^2 + \dots$

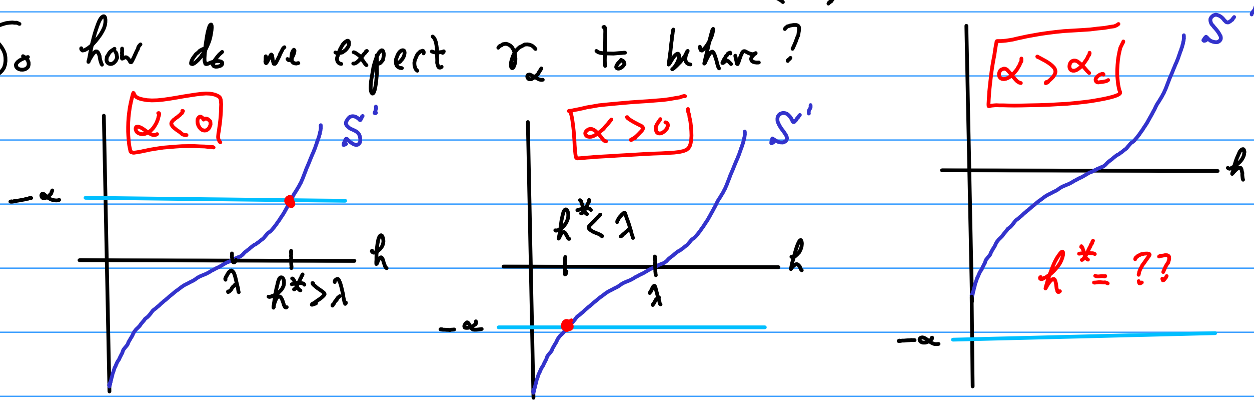
which we use to evaluate the integral. Find:

$$\langle \theta^\alpha \rangle(t) \sim e^{-\gamma_\alpha t}, \quad \text{where} \quad \gamma_\alpha = H(h^*)$$

Note that we <u>do not</u> have $\langle \theta^\alpha \rangle \sim e^{-\alpha \gamma t}$, which would be the case if θ decayed the same pointwise everywhere.

$$\text{Kurtosis} \sim \frac{\langle \theta^\alpha \rangle}{\langle \theta \rangle^\alpha} \sim e^{-\gamma_\alpha t}$$

So how do we expect γ_α to behave?



We have $\gamma_0 = 0$, since $S'(h-\lambda) = 0$ at $h=\lambda$, and $S(0) = 0$.

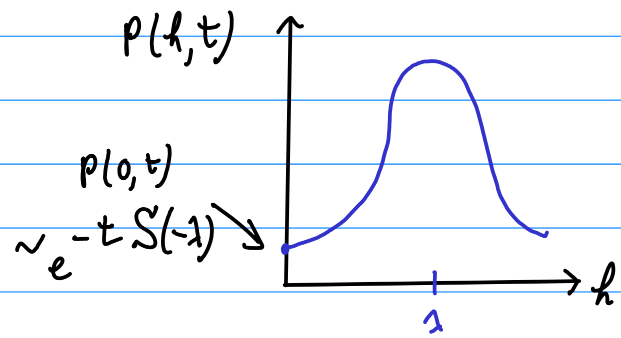
$\hookrightarrow \langle \theta^0 \rangle = 0$ ok!

Hence, γ_α changes sign at $\alpha = 0$.

What happens for $\alpha > \alpha_c$? No saddle point, since would require $h^* < 0$ (not allowed). Hence, take $h^* = 0$ (slowest decay)

The moments $\langle \theta^\alpha \rangle$ decay slower than expected, all the more so for larger α: INTERMITTENCY

Why the leveling-off? For large α, $\langle \theta^\alpha \rangle$ is dominated by realizations with large θ, that is, having experienced little stretching. For $\alpha > \alpha_c$, these are all that matter, so γ_α is the rate of decay of realizations with no stretching,



All this was for realizations of just one blob, but can scale up to many blobs. (See papers quoted) Validity of theory still controversial, but should work for times that are not too long, scales not too large.