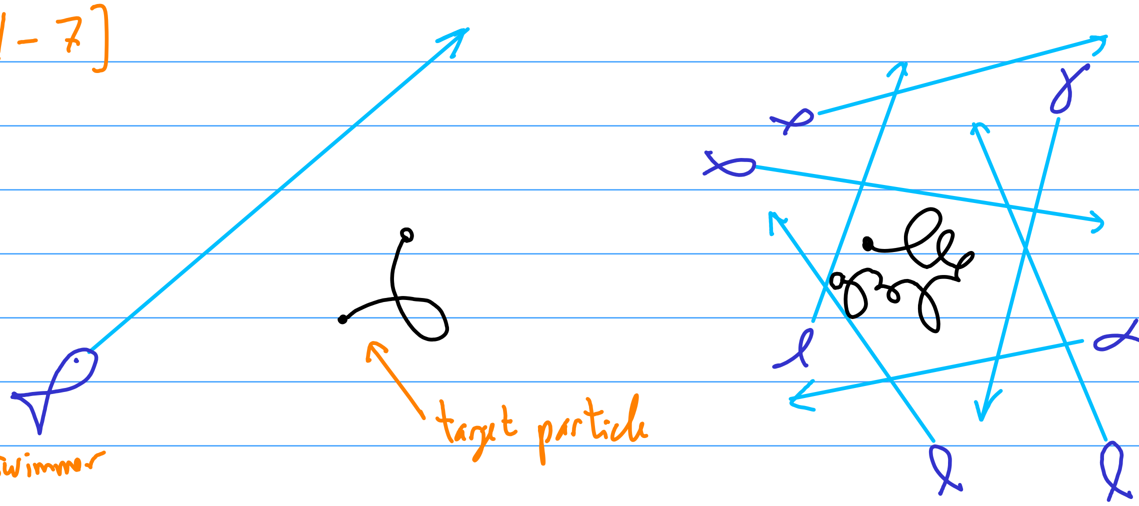
GFD Lectures: Swimming & Swirling

2010/06/23

Lecture 2: Stirring by swimming organisms

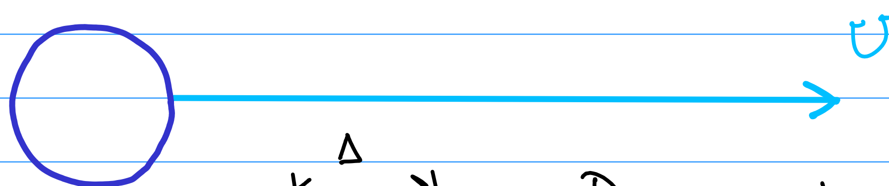
[Slide 1-7]

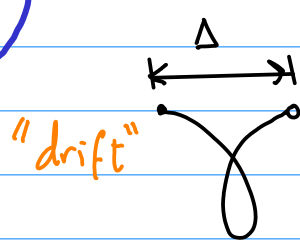
Vision:



Target particle moves under the influence of many swimmers

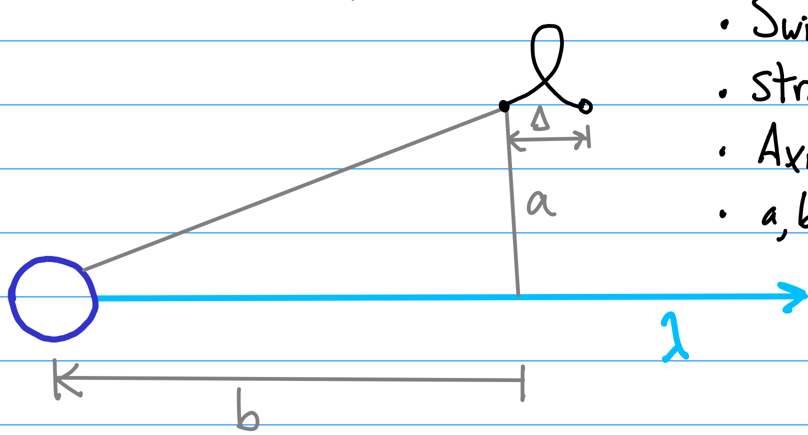
Single swimmer: take a cylinder





Darwin Jr.'s "elastica"
(in potential flow)

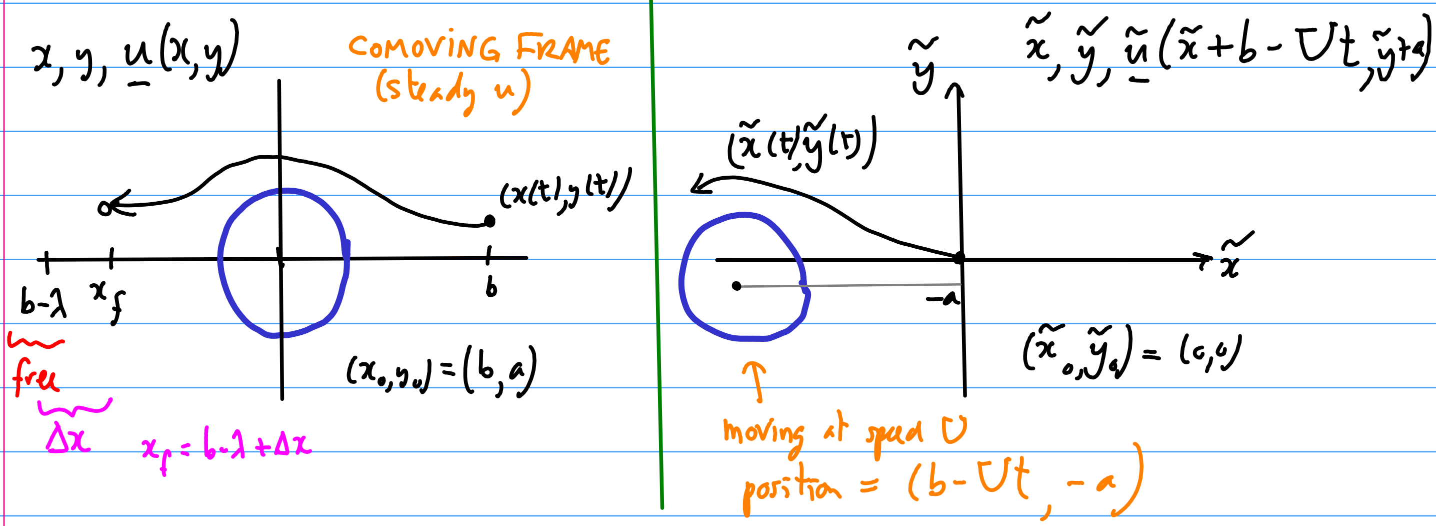
How do we compute Δ?



- Swimming velocity is U (const)
- Straight line for distance λ.
- Axially-symmetric, steady swimmer
- a, b are "impact parameters" $(a>0)$

Compute $\Delta_\lambda(a,b)$

Do 2D case (axisymmetric 3D similar):



$$\frac{d\tilde{x}}{dt} = \tilde{u}(\tilde{x}(t)+b-Ut, \tilde{y}(t)+a), \qquad x = \tilde{x}+b-Ut$$

$$\frac{dx}{dt} + U = \tilde{u}(x,y) \iff \boxed{\frac{dx}{dt} = -U + \tilde{u}(x,y) = u(x,y)}$$

autonomous (better!)

$$-\lambda + \Delta x = \int_0^{T=\lambda/U} u(x(t), y(t))\, dt \qquad \text{For } y: \frac{dy}{dt} = \tilde{v}(x,y)$$

need both

Alternate form: $T = \dfrac{\lambda}{U} = \displaystyle\int_b^{x_f} \frac{dx}{u(x,y)}, \quad x_f = b-\lambda+\Delta x$

how far particle moves when "free-streaming"

$$\frac{\lambda}{U} = \int_b^{b-\lambda+\Delta x} \frac{dx}{u} = -\int_b^{b-\lambda+\Delta x} \frac{dx}{|u|} = \int_{b-\lambda+\Delta x}^{b} dx/|u|$$

$$= \int_{b-\lambda}^{b} \frac{dx}{|u|} + \int_{b-\lambda+\Delta x}^{b-\lambda} \frac{dx}{|u|}$$

If particle doesn't move much and $|b-\lambda|$ "large", then $|u| \simeq U$

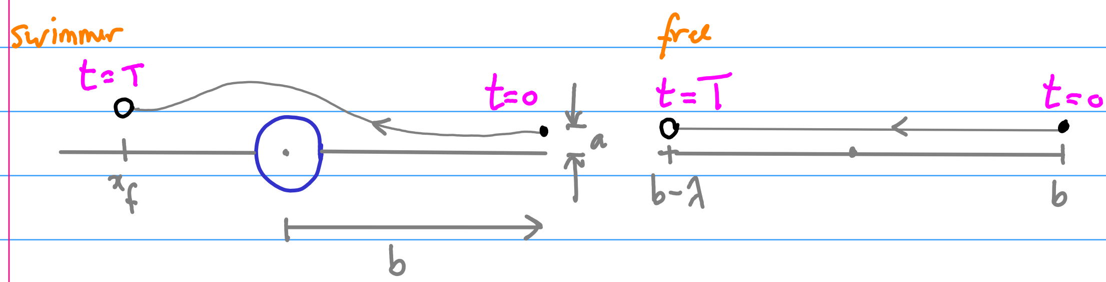
$$\frac{\lambda}{U} \simeq \int_{b-\lambda}^{b} \frac{dx}{|u|} - \frac{\Delta x}{U} \iff \Delta x = \int_{b-\lambda}^{b} \frac{dx}{|u|} - \frac{\lambda}{U}$$

$$\boxed{\Delta x \simeq \int_{b-\lambda}^{b} \left(\frac{1}{|u|} - \frac{1}{U} \right) dx}$$

Better form, since now can take $b \to \infty$, $b-\lambda \to -\infty$ if we want.

"Rayleigh form"

Intuitively, this formula measures the "lag" behind a free-streaming particle:



$$T = \frac{\lambda}{U}$$

2D incompressible: $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$$\psi(x_f, a+\Delta y) = \psi(b, a)$$ Same streamline

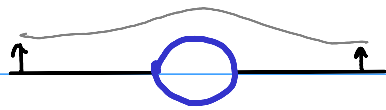
$$\psi(b-\lambda+\Delta x, a+\Delta y) = \psi(b, a)$$ solve for Δy, given Δx

If $|b-\lambda| \gg \Delta x$, $\boxed{\psi(b-\lambda, a+\Delta y) \simeq \psi(b, a)}$ solve for Δy

If also $\Delta y \ll a$, $\psi(b-\lambda, a) + \Delta y \underbrace{\partial_y \psi(b-\lambda, a)}_{u(b-\lambda, a)} \simeq \psi(b, a)$

$$\Delta y \simeq \frac{\psi(b,a) - \psi(b-\lambda, a)}{u(b-\lambda, a)}$$

Now for infinite λ, we have:



$$\Delta y = \frac{\psi(\infty,a) - \psi(-\infty,a)}{U} = 0!$$

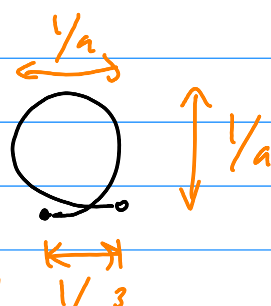
$$\boxed{\Delta y = 0}$$ for $\lambda \to \infty$, $b-\lambda \to -\infty$

Cylinder in potential flow:

$$\psi(x,y) = -Uy\left(1 - \frac{\ell^2}{x^2+y^2}\right)$$ Set $U = \ell = 1$

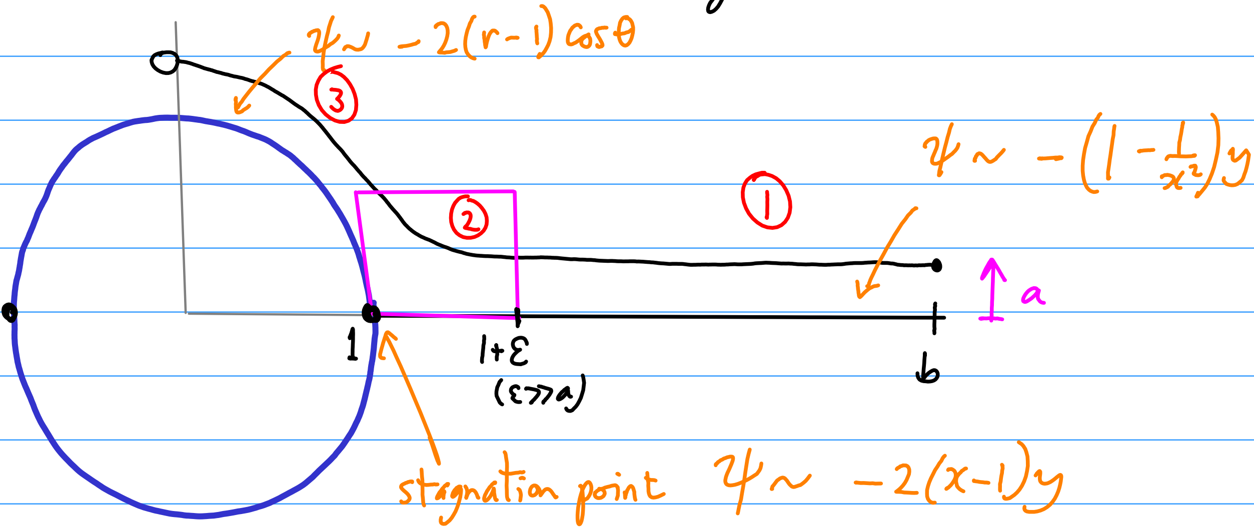
Far away, $\tilde{\psi} \sim \frac{y}{r^2}$, so $\tilde{u} \sim \frac{1}{r^2}$ ← in fixed frame

However, trajectories are almost closed,



Net result is $\Delta(a) \sim \frac{1}{a^3}$ Much smaller than overall excursion!

The limit $a \ll 1$ is more interesting:



free flow ⊃ (to the left)

Need to calculate $\int\left(\frac{1}{u}+1\right)dx$ over each region ①, ②, ③.

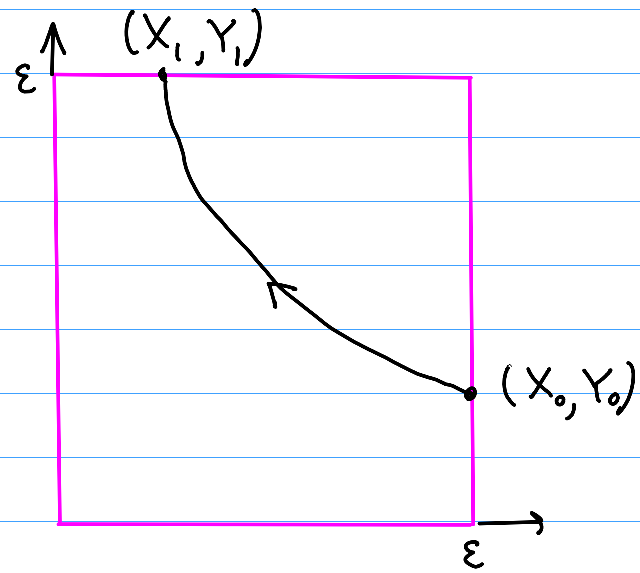
Region 1: $\psi_0 = \psi(b,a) = -(1-b^{-2})a$

$$u = -(1-x^{-2}), \quad T_1 = \int_b^{1+\varepsilon}\left(\frac{1}{u}+1\right)dx = \int_b^{1+\varepsilon}\frac{dx}{1-x^2}$$

transit time

After using $\varepsilon \ll 1$, $b \gg 1$: $T_1 \simeq \frac{1}{2}\log(2/\varepsilon) + \varepsilon/4 - b^{-1} + O(\varepsilon^2, b^{-2})$

Region 2:



$X = x-1$, $Y = y$

$\psi = -2XY$

At X_0, Y_0,

$$\psi = -2\underbrace{X_0}_{\varepsilon}Y_0 = -(1-\overbrace{b^{-2}}^{\text{neglect}})a$$

$$\Rightarrow Y_0 = \frac{a}{2\varepsilon}$$

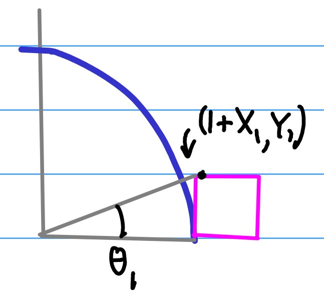
$(X_0, Y_0) = (\varepsilon, a/2\varepsilon)$

But also $Y_1 = \varepsilon$, so $X_1 = a/2\varepsilon$.

$$T_2 = \int_{X_0}^{X_1}\left(\frac{1}{u}+1\right)dx = \int_{\varepsilon}^{a/2\varepsilon}\left(\frac{1}{(-2X)}+1\right)dx = -\frac{1}{2}\log\left(a/2\varepsilon^2\right) + \frac{a}{2\varepsilon} - \overbrace{\varepsilon}^{\text{small}}$$

$$u = -1 + \frac{\cos 2\theta}{r^2}$$

Region 3:



$$T_3 = \int_{\theta_1}^{\pi/2} \left(\frac{1}{u} + 1\right) \underbrace{\frac{dx}{d\theta}}_{-r\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{\theta_1}^{\pi/2} \frac{\cos 2\theta}{\sin\theta}\, d\theta \qquad , \ \theta_1 \text{ small.}$$

$$T_3 \simeq -1 + \frac{1}{2}\log 2 - \frac{1}{2}\log\theta_1 + O(\theta_1^2)$$

$$\tan\theta_1 = \frac{Y_1}{1+X_1} = \frac{\varepsilon}{1+a/\varepsilon} \simeq \varepsilon(1 - a/\varepsilon) = \varepsilon - a$$

$$\therefore\ T_3 \simeq -1 + \frac{1}{2}\log 2 - \frac{1}{2}\log\varepsilon + \frac{1}{2}\frac{a}{\varepsilon} + O\left((a/\varepsilon)^2\right)$$

Add everything together:

divergent $\log\varepsilon$ terms cancel

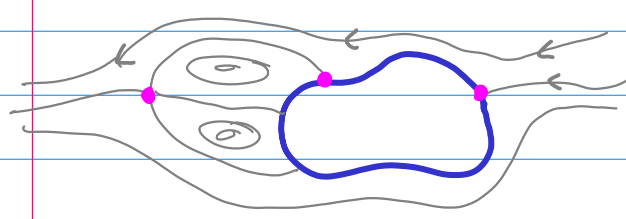
$$T = T_1 + T_2 + T_3 = \left(\frac{1}{2}\log(2/\cancel{\varepsilon}) - 6^{-1}\right) + \left(-\frac{1}{2}\log\left(\frac{a}{2\cancel{\varepsilon}}\right)\right) + \left(-1 + \frac{1}{2}\log 2 - \cancel{\frac{1}{2}\log\varepsilon}\right)$$

$$T = \underbrace{-\frac{1}{2}\log a}_{} - 1 + \frac{3}{2}\log 2 - 6^{-1}$$ to leading order.

Dominant term for small a.
Comes only from region 2, near stagnation point.

$T \to \infty$ for $a \to 0$. particle gets stuck!

The total drift is given by $2T$, since the body is fore-aft symmetric.



In general, the coefficient of $\log a$ is given by summing over the linearization coeffs for each (hyperbolic) stagnation pt encountered. (not true for no-slip!)

Note that to pick up the $-\log a$ contribution, the target particle must come in the vicinity of the stagnation points

$$\Delta_\lambda(a,b) = \begin{cases} -\log a \quad , & 0 \leq b \leq \lambda \\ \text{(neglect)} \quad , & \text{otherwise} \end{cases}$$

Effective diffusivity:

What we have: $\Delta_\lambda(a,b)$ Need: effective diffusivity

Constants: U, ℓ, λ, n Random: a, b

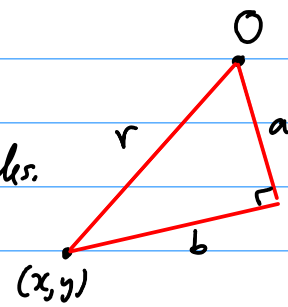
number density

If we pick a random point in space, what is PDF of a, b?

$$\frac{1}{V}\,dx\,dy \rightarrow p(a,b)\,da\,db$$

volume

Assume target particle at origin:



- Hard way: compute (a,b) from (x,y), transform variables.
- Easier: note (a,b) just like (x,y), but rotated, and $a>0$.

irrelevant, by isotropy

Hence: $\frac{1}{V}\,dx\,dy = \frac{1}{V}\,2\,da\,db$ 2D

3D

In 3D, $\frac{1}{V} dx\, dy\, dz = \frac{1}{V} 2\pi a\, da\, db$ ← like cylindrical coordinates, integrated over θ.

Now, assume target particle is "kicked" by swimmer:

$$\underline{x}_N = \underbrace{\underline{x}_0}_{0} + \sum_{k=1}^{N} \Delta_\lambda(a_k, b_k)\,\hat{r}_k$$

$a_k, b_k, \hat{r}_k$ random independent & identical

On average, particle goes nowhere: $\langle \underline{x}_N \rangle = 0$

$$\langle |\underline{x}_N|^2 \rangle = \sum_{k=1}^{N} \langle \Delta_\lambda^2(a_k, b_k)\, \hat{r}_k \cdot \hat{r}_k \rangle + \text{vanishing cross terms } \langle \hat{r}_k \rangle = 0$$

$$= N \langle \Delta_\lambda^2(a,b) \rangle$$

$$= \frac{N}{V} \int \Delta_\lambda^2(a,b)\, 2\, da\, db \quad \text{2D}$$

What is N? # of "collisions" $N = t/T$ ← elapsed time; T ← "mean free time"

$T = \lambda/U$ λ = mean free path

Hence, $\langle |x(t)|^2 \rangle = \frac{Ut}{\lambda} \underbrace{\frac{1}{V}}_{\text{what about this?}} \int \Delta_\lambda^2(a,b)\, 2\, da\, db$ $x_N \to x(t)$

Only one swimmer, so $\frac{1}{V} = n$, the number density

$$\langle |x(t)|^2 \rangle = 2\frac{Unt}{\lambda} \int \Delta_\lambda^2(a,b)\, da\, db = 2d\kappa t = 4\kappa t$$

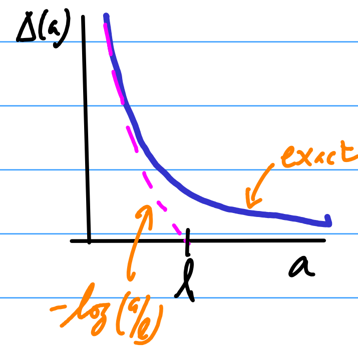
↑ dimension of space ↑ effective diff

This depends on integral of squared displacement. Actual mass displaced could be 0!

$$\boxed{\kappa = \begin{cases} \dfrac{Un}{2\lambda} \displaystyle\int \Delta_\lambda^2(a,b)\, da\, db & \text{2D} \\[2ex] \dfrac{\pi U n}{3\lambda} \displaystyle\int \Delta_\lambda^2(a,b)\, a\, da\, db & \text{3D} \end{cases}}$$

effective diffusivity

Recall our approximate form for cylinder: $\Delta_\lambda(a,b) = \begin{cases} \overset{\ell}{-\log a}, & 0 \leq b \leq \lambda \\ 0, & \text{otherwise} \end{cases}$



Cylinder. $\kappa \simeq \dfrac{2Un}{\cancel{\lambda}} \cancel{\lambda} \displaystyle\int_0^\ell \log^2\left(\frac{a}{\ell}\right) da$

$$\int \log^2 x\, dx = x \log^2 x - 2x \log x + 2x$$

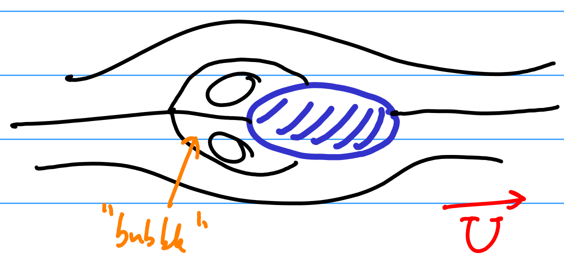
$$\int_0^1 \log^2 x\, dx = 2$$ (numerical answer: 2.37)

$$\boxed{\kappa \simeq U n \ell^3}$$ (numerical: $\kappa = 1.19\, U n \ell^3$)

Note that this is completely independent of λ!

$\Rightarrow$ see computer simulations

Another example: consider a swimmer with a bubble "wake":



If a particle is trapped in the bubble, moves by λ.

$$\Delta_\lambda(a,b) = \begin{cases} \lambda, & \text{particle inside bubble} \\ \text{neglect}, & \text{otherwise} \end{cases}$$

Total volume of bubble

$$6\kappa = \frac{2Un}{\cancel{\lambda}} \int_{\text{inside bubble}} \lambda^{\cancel{2}}\, da\, db = Un\lambda V_{bubble}$$

The 2 goes away since 2dadb is volume element

$$\boxed{\kappa = \frac{1}{6} Un\lambda V_{bubble}}$$

V_{bubble} = area in 2D
= volume in 3D

Now this depends on path length λ. This can be much larger than for untrapped fluid. Real swimmer probably in between

(Viscous swimmer with boundary layer: $\kappa \sim \log\lambda$)

swimmer	λ-dependence	far/near field dominance
potential (slip)	none	near
viscous (slip) squirmer	none	far
viscous (no-slip)	$\log\lambda$	near
trapped	λ	near

More topics:
- Green-Kubo
- Wakes
- Far field
- Levy flights
- Stratification