GFD Lectures: Swimming & Swirling

2010/06/21

Lecture 1: Stirring & Mixing

Stirring: mechanical action (cause)
Mixing: homogenization of a scalar (effect)

$\theta(\underline{x}, t)$ = concentration, $\underline{u}(\underline{x}, t)$ given

Advection Diffusion eq. (AD)

$$\frac{\partial \theta}{\partial t} + \underline{u}\cdot\nabla\theta = \kappa \nabla^2 \theta, \quad \nabla\cdot\underline{u} = 0 \quad \text{in } \Omega$$

Boundary conditions: $\left.\begin{array}{l}\hat{n}\cdot\nabla\theta = 0 \\ \hat{n}\cdot\underline{u} = 0\end{array}\right\}$ on boundary $\partial\Omega$

Let $\langle \cdot \rangle = \int_\Omega \cdot \, dV$

Multiply AD by $m\,\theta^{m-1}$, integrate:

$$\langle m\theta^{m-1}\partial_t\theta\rangle = \partial_t\langle\theta^m\rangle$$

$$\langle m\theta^{m-1}\underline{u}\cdot\nabla\theta\rangle = \langle \underline{u}\cdot\nabla\theta^m\rangle \overset{\nabla\cdot\underline{u}=0}{=} \langle\nabla\cdot(\underline{u}\theta^m)\rangle$$

$$= \int_{\partial\Omega} \theta^m \underbrace{\underline{u}\cdot\hat{n}}_{0!}\, dS = 0$$

$$\langle m\theta^{m-1}\kappa\nabla^2\theta\rangle = \kappa m\,\langle \nabla\cdot(\theta^{m-1}\nabla\theta) - \nabla\theta^{m-1}\cdot\nabla\theta\rangle$$

$$= \kappa m \int_{\partial\Omega} \theta^{m-1} \underbrace{\nabla\theta\cdot\hat{n}}_{0!}\, dS - \kappa\, m(m-1)\,\langle \theta^{m-2}|\nabla\theta|^2\rangle$$

$$\boxed{\partial_t \langle \theta^m \rangle = -\kappa\, m(m-1) \langle \theta^{m-2} |\nabla\theta|^2 \rangle}$$

$m=0$ is trivial

$m=1$: $\partial_t \langle \theta \rangle = 0$ Total amount of θ is conserved

$m=2$: $\partial_t \langle \theta^2 \rangle = -2\kappa \langle |\nabla\theta|^2 \rangle$ $\langle \theta^2 \rangle$ non-increasing!

Let variance $\mathrm{Var} = C_2 = \langle \theta^2 \rangle - \underbrace{\langle \theta \rangle^2}_{\text{constant}}$

$$\partial_t C_2 = -2\kappa \langle |\nabla\theta|^2 \rangle$$

Scenario:

- Variance can only decrease.
- Slows down as $\langle |\nabla\theta|^2 \rangle \to 0$
- But $\langle |\nabla\theta|^2 \rangle = 0$ iff $\theta = \underbrace{\text{constant}}_{\text{in some sense}}$.

Hence the system is "driven" towards a homogeneous state where

$$\theta(\underline{x}, t) = \langle \theta \rangle = \text{constant.} \qquad (C_2 = 0,\ \langle \theta^2 \rangle = \langle \theta \rangle^2)$$

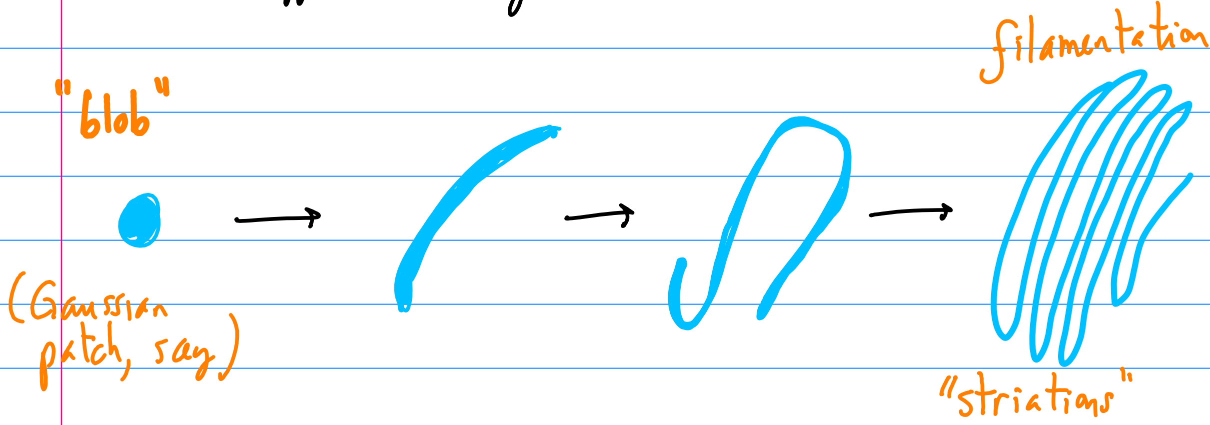
Assume $\langle \theta \rangle = 0$ WLOG

No fluctuations from the mean! When C_2 is small "enough", we say the system is mixed.

Big Q: Where is $\underline{u}(\underline{x}, t)$!? (stirring)

It doesn't appear in the variance equation!

But of course the variance equation is not closed: it depends on $\nabla\theta$. What happens when you stir?



This hints at the answer: stirring increases $\nabla\theta$

$$\partial_t \langle \theta^2 \rangle = -2\kappa \underbrace{\langle |\nabla\theta|^2 \rangle}_{\text{this becomes larger as we stir}}$$

By how much are gradients increased? After all, if $|\nabla\theta|$ becomes too large, then $\langle \theta^2 \rangle \to 0$, so there are no gradients anymore!

Answer: for "good" stirring, the system is driven to a state where

$$\kappa \langle |\nabla\theta|^2 \rangle \longrightarrow \boxed{\text{independent of } \kappa}$$

Hence, $\boxed{\nabla\theta \sim \kappa^{-1/2}}$

This is the chaotic/turbulent mixing scenario:

$\dfrac{\partial \langle \theta^2 \rangle}{\partial t}$ becomes independent of κ after a "short" transient

(How short? Typically $\sim \log \kappa$)

This is the Platonic ideal of mixing

Furthermore, the smallest scales visible in the concentration field $\theta(\underline{x}, t)$ have size $\sim \sqrt{\kappa}$. (missing a dimensional factor → see later)

Note that $\partial_t\langle\theta^2\rangle$ independent of κ is crucial: in most applications, κ is tiny!

Heat: $\kappa = 2.2160 \times 10^{-5}\ m^2/s$ at 300K

10 m room: diffusion time $\sim \frac{L^2}{\kappa} = \frac{(10m)^2}{(2\times10^{-5} m^2/s)} \sim 4.5 \times 10^6$ sec

~ 1300 hours

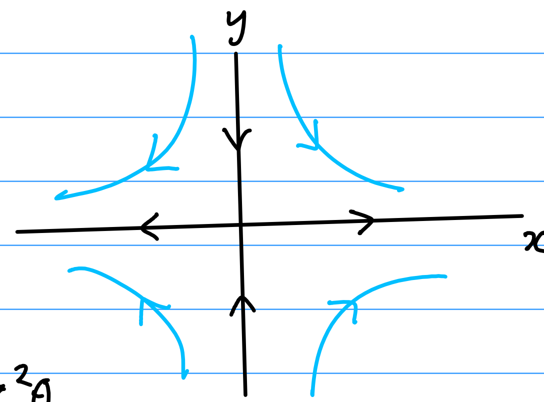
$\boxed{\sim 53 \text{ days !}}$

So we better stir!
Even thermal convection is often enough.

Example of a good mixer:

$$\underline{u}(\underline{x}, t) = (\lambda x, -\lambda y)$$

"hyperbolic point"



AD: $\partial_t\theta + \lambda x \partial_x\theta - \lambda y \partial_y \theta = \kappa \nabla^2\theta$

Can solve this exactly (we'll say more next time), but let's do the simplest thing: look for an x-independent solution of the form:

$$\theta(\underline{x}, t) = e^{-\lambda t} f(y)$$

$$-\lambda f - \lambda y f' = \kappa f''$$

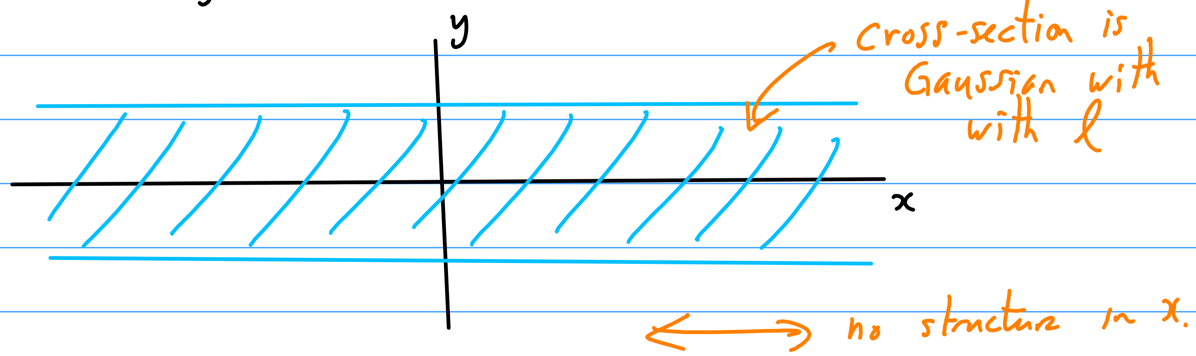
Boundary condition:

$f \to 0$ as $y \to \pm\infty$.

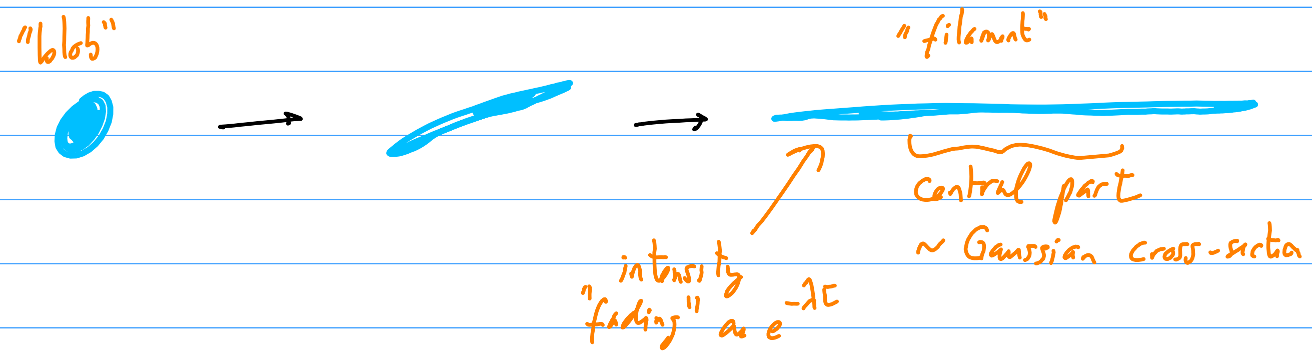
Solution is: $f(y) = e^{-y^2/2\ell^2}$, where $\ell^2 = \frac{\kappa}{\lambda}$

Hence,
$$\boxed{\theta(\underline{x}, t) \sim e^{-\lambda t} e^{-y^2/2\ell^2}}$$

This is the "filament" solution:



In fact, this solution tells us about the ultimate state of any compactly-supported initial condition:



For this case, we know the length scale of "striations":

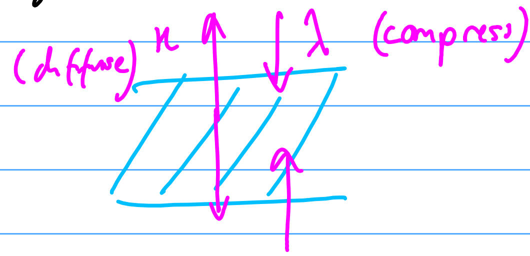
$$\boxed{\ell = \sqrt{\frac{\kappa}{\lambda}}}$$

Batchelor length

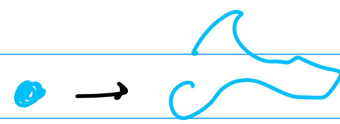
Note $\ell \sim \sqrt{\kappa}$, as necessary to make decay rate indep. of κ!

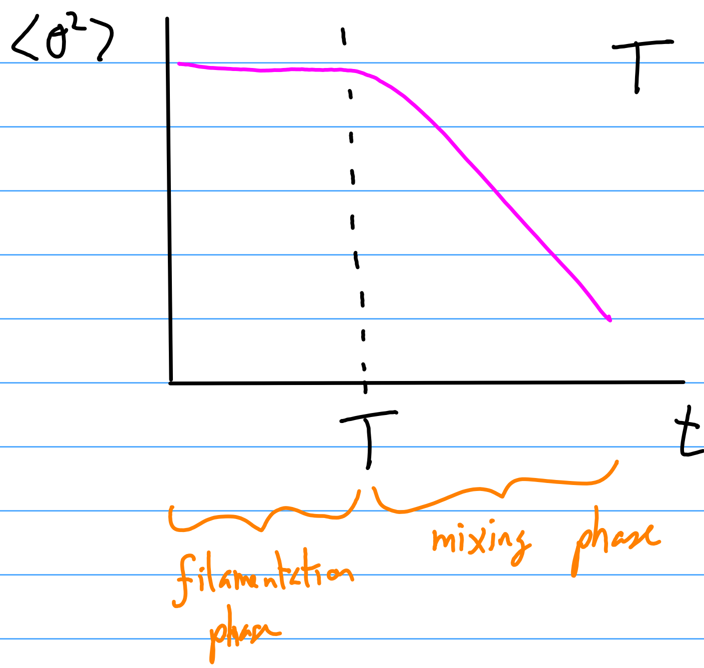
In practical applications, λ is often taken to be the local rate of strain.

l is set by a balance between compression and diffusion



Summary: how mixing proceeds

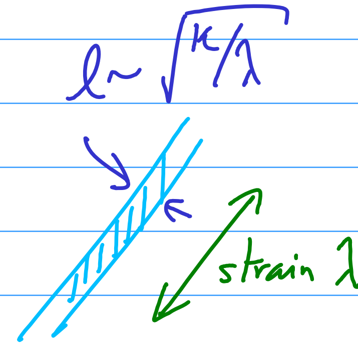
- A blob is stirred 
- For a while, $\langle \theta^2 \rangle$ is $\sim$ constant, since κ is small
- When $\nabla\theta$ reaches scales of order l, diffusion takes over
- After that, $\langle \theta^2 \rangle$ decays at a κ-independent rate



T given by: $e^{-\lambda T} \sim \sqrt{\kappa}$

$$T \sim \lambda^{-1} \log \kappa$$

Effective Diffusivity



Recall: filaments in chaotic advection

Goal was to compute decay of variance,

$$\langle \theta^2 \rangle \sim e^{-\gamma t} \qquad (\gamma = \lambda \text{ for uniform strain})$$

But when can we replace the advection-diffusion equation by an "effective" diffusion equation?

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad \Longrightarrow \quad \frac{\partial \theta}{\partial t} = K_{eff} \nabla^2 \theta \ ?$$

Diffusion arises from noise: $x_n = x_{n-1} + \xi_n$

i.i.d. (Gaussian)

Assume $\langle \xi_n \rangle = 0$, $\langle \xi_n^2 \rangle = \sigma^2$

$$x_n = \underbrace{x_0}_{0} + \sum_{i=1}^{n} \xi_n , \qquad \langle x_n \rangle = 0$$

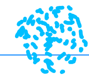
$$\langle x_n^2 \rangle = \sum_{i=1}^{n} \langle \xi_n^2 \rangle = \overset{\sim \text{"time"}}{n} \sigma^2 = \underbrace{2Kt}_{\text{by definition}}$$

$t = nT$

In d dimensions,

$$\langle x_n^2 + y_n^2 (+ z_n^2) \rangle = n d \sigma^2 = 2dKnT$$

$$\boxed{K_{eff} = \frac{\sigma^2}{2T}}$$

Now if we take a "cloud" of points , and define a density

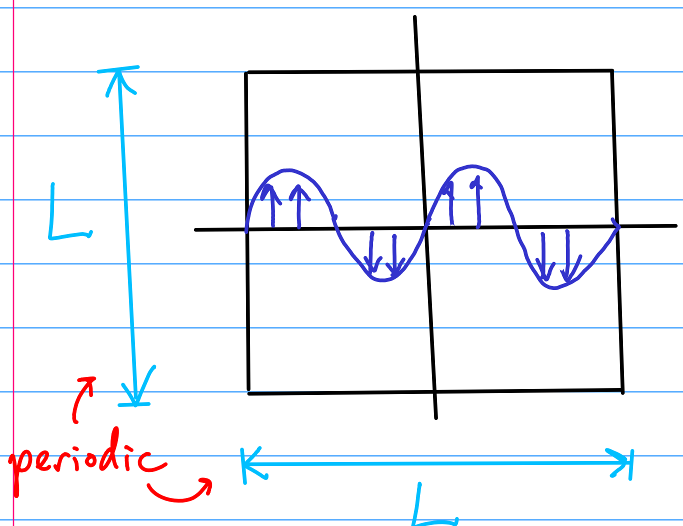
$$\theta(\underline{x}, t) = \text{density of points}$$

Then θ satisfies $\dfrac{\partial \theta}{\partial t} = K \nabla^2 \theta$ if each point evolves independently according to $x' = x + \xi$.

Of course, this requires "coarse-graining": it is only true if we don't look to closely (scales $\lesssim \sigma$) or too often (time scales $\lesssim T$).

This provides clues as to when the concept of an effective diffusivity makes sense.

Rest of lecture: look at an example, the famous SINE FLOW.



- Velocity field (shear flow)

$$\underline{u}_H = \left(U \sin\left(\frac{2\pi k y}{L}\right), 0 \right)$$ STEP 1

applied for $0 \leq t < \tau/2$

- $$\underline{u}_V = \left(0, U \sin\left(\frac{2\pi k x}{L}\right) \right)$$ STEP 2

for $\tau/2 \leq t < \tau$.

Can solve $\dot{\underline{x}} = \underline{u}$, $\underline{x}(0) = \underline{x}_0$ exactly:

STEP 1: $$x(\tau/2) = x_0 + U\tau/2 \sin\left(\frac{2\pi k y_0}{L}\right)$$

$$y(\tau/2) = y_0$$

STEP 2: $x(\tau) = x(\tau/2)$

$x(\tau/2) = x(\tau)$

$$y(\tau) = y(\tau/2) + \frac{U\tau}{2}\sin\left(\frac{2\pi k x(\tau/2)}{L}\right)$$

Write as one map of period τ:

$$\boxed{\begin{aligned} x' &= x + T\sin(2\pi k y/L) \\ y' &= y + T\sin(2\pi k x'/L) \end{aligned}}$$

$$T \equiv \frac{U\tau}{2}$$

Easy to iterate on a gazillion particles.

note x'! Important for area-preservation (comes from incompressibility)

Example 1: Run Matlab script example(1).

$$L = k = 1, \quad T = 0.1$$

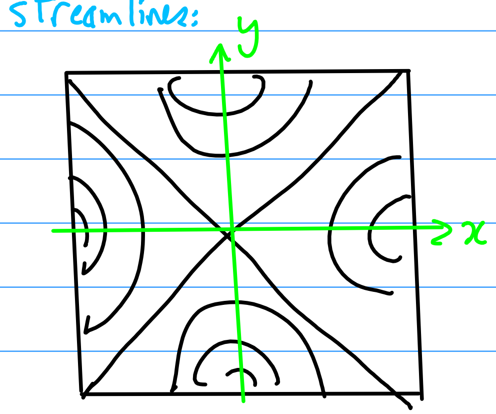
Note how regular the orbits are: for small T the map is effectively a symplectic integrator

$$\frac{x'-x}{T} = \sin\left(\frac{2\pi k y}{L}\right), \quad \frac{y'-y}{T} = \sin\left(\frac{2\pi k x'}{L}\right)$$

As $T \to 0$, this approximates $\frac{dx}{dt} = \sin\left(\frac{2\pi k y}{L}\right) = \partial\psi/\partial y$, $\frac{dy}{dt} = \sin\left(\frac{2\pi k x}{L}\right) = -\partial\psi/\partial x$,

or flow with streamfunction: $\psi = \frac{L}{2\pi k}\left(\cos\left(\frac{2\pi k x}{L}\right) - \cos\left(\frac{2\pi k y}{L}\right)\right)$

streamlines:



The streamlines aren't traced exactly because T is finite.

Example 2 adds a bit of noise.

$$x' = (\text{sine map}) + \sqrt{2D}\,\xi$$

Gaussian random var. with $\langle \xi^2 \rangle = 1$.

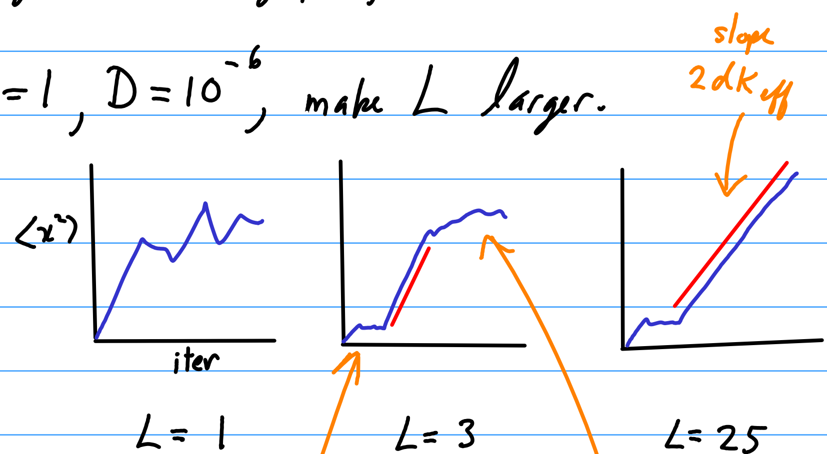
Example 3: $T = 1$. Now doesn't approximate a flow at all $\rightarrow$ CHAOTIC.

Example 4: $T = 1$, $L = 1$, $D = 10^{-4}$: "fat" filaments.
- $\rightarrow$ measure width by clicking
- $\rightarrow$ repeat for $D = 10^{-6}$
- $\rightarrow$ observe rough $\sqrt{D}$ scaling for filament width

(see Lecture 1)

Example 5: $T = 1/2$, $k = 1$, $D = 10^{-6}$, make L larger.

Plot $\langle x^2 \rangle$ vs iteration



Hence, the concept of an effective diffusivity makes sense if we look at large scales, such that we cannot see the correlated small scale motions, and long times. (but not too long!)

$\rightarrow$ Useful for turbulence

$$K_{eff} \sim 0.068 \gg D = 10^{-6}$$

Note that the "cross" shape  evident in the pattern is not captured.