Lecture 34: Permutations generated by Brownian particles

1 Brownian particles on the real line

Consider n Brownian particles on the real line, with diffusion constant D. The position of each particle is denoted $x_k(t)$. The position of a walker at time t has a probability density function $p(x, t; x', 0)$ that satisfies the heat equation,

$$\partial_t p - D\Delta p = \delta(x - x')\,\delta(t), \tag{1}$$

where x' is the initial position of the particle, with solution

$$p(x, t; x', 0) = \frac{1}{\sqrt{4\pi Dt}}\,\mathrm{e}^{-(x-x')^2/4Dt}\,, \qquad t > 0. \tag{2}$$

Assume the initial ordering

$$x_1(0) < x_2(0) < \cdots < x_n(0), \tag{3}$$

and define the probability $P(t, s)$ that the particles are ordered according to the permutation $s \in S_n$, the symmetric group on n symbols; thus,

$$P(0, s) = \begin{cases} 1, & s = \text{id}; \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

At later times, we have

$$P(t, s^{-1}) = \int_{-\infty}^{\infty} \mathrm{d}x_{s(1)} \int_{x_{s(1)}}^{\infty} \mathrm{d}x_{s(2)} \cdots \int_{x_{s(n-1)}}^{\infty} \mathrm{d}x_{s(n)} \prod_{k=1}^{n} p(x_{s(k)}, t; x_{s(k)}(0), 0)\,. \tag{5}$$

That is, the particle $s(1)$ is to the left of all the others, $s(2)$ is to the right of $s(1)$ but to the left of the rest, etc.

Now let

$$y_k := x_{s(k)}/\sqrt{4Dt}, \qquad \varepsilon_k(t) := x_{s(k)}(0)/\sqrt{4Dt}\,, \tag{6}$$

from which (5) becomes

$$P(t, s^{-1}) = \int_{-\infty}^{\infty} \mathrm{d}y_1 \int_{y_1}^{\infty} \mathrm{d}y_2 \cdots \int_{y_{n-1}}^{\infty} \mathrm{d}y_n \prod_{k=1}^{n} \frac{1}{\sqrt{\pi}}\, \mathrm{e}^{-(y_k - \varepsilon_k(t))^2}\,. \tag{7}$$

For large time, we have $\varepsilon_k(t) \ll 1$, and

$$\mathrm{e}^{-(y_k - \varepsilon_k(t))^2} = \mathrm{e}^{-y_k^2}\,(1 + 2\varepsilon_k y_k) + \mathrm{O}(\varepsilon^2). \tag{8}$$

We must have

$$\frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \mathrm{d}y_1 \int_{y_1}^{\infty} \mathrm{d}y_2 \cdots \int_{y_{n-1}}^{\infty} \mathrm{d}y_n\, \mathrm{e}^{-\sum_{k=1}^{n} y_k^2} = \frac{1}{n!}. \tag{9}$$

(This must work for any PDF with the right symmetry, so must be the fraction of volume occupied by an n-dimensional 'wedge.')

Define

$$I_{n,\ell} := -\frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \mathrm{d}y_1 \int_{y_1}^{\infty} \mathrm{d}y_2 \cdots \int_{y_{n-1}}^{\infty} \mathrm{d}y_n\, y_\ell\, \mathrm{e}^{-\sum_{k=1}^{n} y_k^2}. \tag{10}$$

By changing order of integration and replacing y_k by $-y_{n-k+1}$, we can show $I_{n,\ell} = -I_{n,n-\ell+1}$. Some specific values are $I_{2,1} = -I_{2,2} = 1/(2\sqrt{2\pi})$, $I_{3,1} = -I_{3,3} = 1/(4\sqrt{2\pi})$, $I_{3,2} = 0$. Challenge: compute this in general (must be known. . .).

In any case, the time-asymptotic solution is

$$P(t, s^{-1}) = \frac{1}{n!} - \sum_{k=1}^{n} 2\varepsilon_k I_{n,k} + \mathrm{O}(\varepsilon^2), \tag{11}$$

which, from (6), shows a rather slow approach to the uniform distribution as $1/\sqrt{t}$.

2 Brownian particles on the unit interval

Now we turn to Brownian particles on the interval $[0, 1]$, with reflecting boundary conditions at the endpoints (Figure 1). The same heat equation (2) is satisfied by the

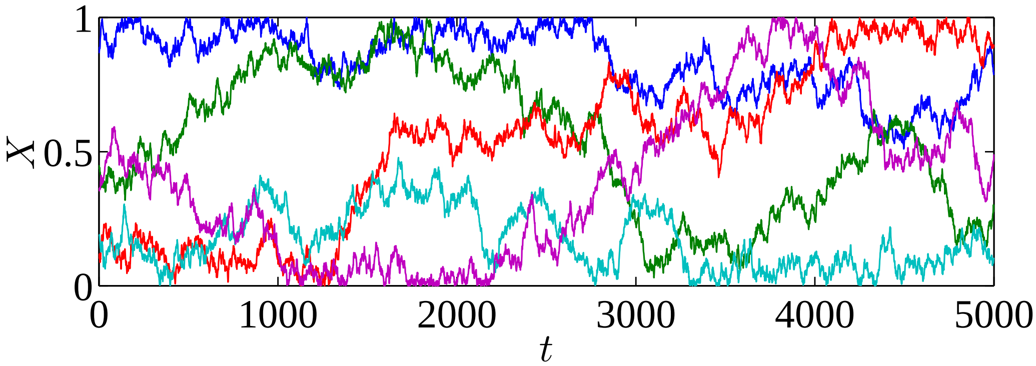

Figure 1: Five Brownian particles on the interval $[0,1]$, with reflecting boundary conditions.

probability density (Green's function), but now the reflecting (Neumann) boundary conditions lead to

$$p(x,t;x',0) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} \left(\mathrm{e}^{-(x-x'-2k)^2/4Dt} + \mathrm{e}^{-(x+x'-2k)^2/4Dt} \right), \quad t > 0, \tag{12}$$

which can also be written

$$p(x,t;x',0) = \sum_{k=-\infty}^{\infty} \cos(\pi k x) \cos(\pi k x') \, \mathrm{e}^{-(\pi k)^2 Dt}, \qquad t > 0. \tag{13}$$

The same formula (5) applies for $P(t, s^{-1})$. Let's take $n = 2$; then after some integrals

$$P(t,s^{-1}) = \frac{1}{2} + {\sum_{\substack{k,\ell \\ k \neq \ell}}}' \frac{(1-(-1)^{k+\ell})}{\pi^2(k^2-\ell^2)} \cos(\pi k x_{s(1)}(0)) \cos(\pi k x_{s(2)}(0)) \, \mathrm{e}^{-\pi^2(k^2+\ell^2)Dt}. \tag{14}$$

($\sum'$ means $k \neq 0$ and $\ell \neq 0$.) The slowest exponential has $k^2 + \ell^2 = 1$; hence, we have

$$\left| P(t,s^{-1}) - \tfrac{1}{2} \right| \leq |C| \, \mathrm{e}^{-\pi^2 Dt} \tag{15}$$

where

$$C = {\sum_{\substack{k,\ell \\ k \neq \ell}}}' \left| \frac{(1-(-1)^{k+\ell})}{\pi^2(k^2-\ell^2)} \right|, \tag{16}$$

as long as C is finite. (Challenge: does this diverge? If so need to refine the analysis.)

In any case it appears to be the right bound: see Fig. 2. Is this a cut-off? How do we show this?

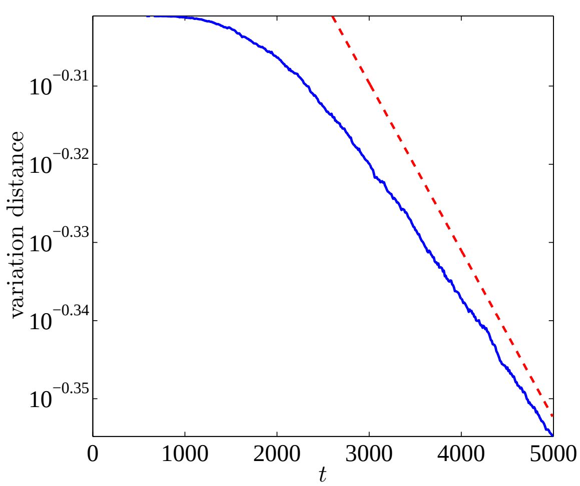

Figure 2: Variation distance as a function of time for 2 particles, with $D = 5 \times 10^{-6}$ (100,000 realizations). The dashed line is proportional to $e^{-\pi^2 D t}$.