Lecture 29: Mix-norms*

1 Norms

In this lecture we define a measure of mixing that does not necessarily require diffusion to measure the amount of homogenization that occurs during the mixing process. Recall the advection-diffusion equation

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = \kappa \nabla^2 \theta, \tag{1}$$

where θ is a concentration field in a finite domain Ω, with no-net-flux boundary conditions. We assume without loss of generality that

$$\int_\Omega \theta \,\mathrm{d}\Omega = 0, \tag{2}$$

and define the L^2-norm, or variance, as

$$\|\theta\|_2^2 = \int_\Omega \theta^2 \,\mathrm{d}\Omega. \tag{3}$$

Recall from Lecture 1 that the variance evolves according to

$$\frac{d}{dt}\|\theta\|_2^2 = -2\kappa \|\boldsymbol{\nabla}\theta\|_2^2, \tag{4}$$

and decays in time as the system mixes. The variance indicates the extent to which the concentration has homogenized and is thus a good measure of the amount of mixing that has occurred. However, the variance requires knowledge of small scales in θ, which we are not necessarily interested in. A measure of how well-mixed the concentration is does not necessarily require knowledge of how much homogenization has occurred due to diffusion at small scales. This is more in keeping with the definition of mixing in the sense of ergodic theory [2]. In this regard, we proceed to consider the pure advection equation

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = 0. \tag{5}$$

Note that in this case equation (4) predicts that the variance satisfies

$$\frac{d}{dt}\|\theta\|_2^2 = 0, \tag{6}$$

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and cannot therefore be used as a measure of mixing.

The advection equation (5) takes us closer to the ergodic sense of mixing in which we think of the advection due to the velocity field as a time-dependent operator $S^t : \Omega \to \Omega$ that moves an initial patch of dye according to

$$\theta_0(\boldsymbol{x}) \mapsto \theta(\boldsymbol{x}, t) = S^t \theta_0(\boldsymbol{x}). \tag{7}$$

If we consider a region A of uniform concentration defined by

$$\theta_0(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in A, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

then the volume of the patch

$$\mathrm{Vol}[\theta(\boldsymbol{x}, t)] = \mathrm{Vol}(A), \tag{9}$$

remains constant in time by incompressibility. We can associate the volume of the patch with the Lebesgue measure and, because of (9), S^t is measure-preserving.

We define mixing in the sense of ergodic theory by

$$\lim_{t \to \infty} \mathrm{Vol}[A \cap S^t(B)] = \mathrm{Vol}(A)\mathrm{Vol}(B), \tag{10}$$

for all patches $A, B \in \Omega$. This definition follows our intuition for what good mixing is. Referring to figure 1, when the system is well-mixed the intersection of A and $S^t B$ is proportional to both $\mathrm{Vol}(A)$ and $\mathrm{Vol}(B)$. Thus, if the condition (10) holds then S^t must spread any initial patch throughout the domain. This condition is referred to as *strong mixing* and can be shown to imply ergodicity.

The intersection of the advected patch B with the reference patch A is analogous to projection onto L^2 functions. This motivates the following *weak convergence* condition

$$\lim_{t \to \infty} \langle \theta(\boldsymbol{x}, t), g \rangle = 0, \tag{11}$$

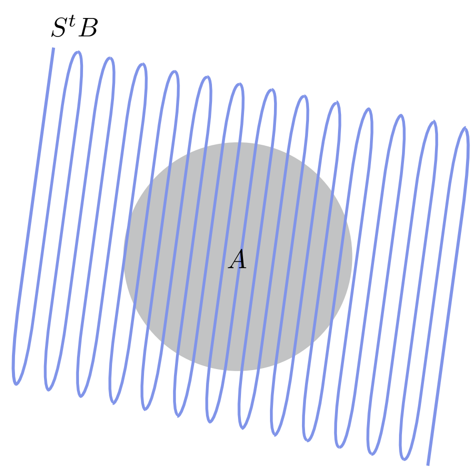

Figure 1: An advected patch $S^t B$ that has undergone strong mixing. At late times the patch covers an arbitrary reference patch A.

for all functions $g \in L^2(\Omega)$, where the inner product is defined by

$$\langle f, g \rangle = \int_\Omega f(\boldsymbol{x}) g(\boldsymbol{x}) \, \mathrm{d}\Omega, \tag{12}$$

and $f \in L^2(\Omega)$ if $\int_\Omega |f|^2 \, \mathrm{d}\Omega < \infty$. Weak convergence is equivalent to mixing as a consequence of the Riemann–Lebesgue lemma. The equivalent conditions (10) and (11) require computing over all patches A or functions g, respectively. Thus, neither of these conditions is useful in practice. However, we proceed to describe a theorem that shows there is a simpler way to determine whether or not weak convergence is satisfied.

Mathew, Mezic and Petzold [5] introduced the mix-norm, which for mean-zero functions is equivalent to

$$\|\theta\|_{\dot{H}^{-1/2}} := \|\nabla^{-1/2}\theta\|_2 \, . \tag{13}$$

Doering and Thiffeault [1] and Lin, Thiffeault and Doering [3] generalized the mix-norm to

$$\|\theta\|_{\dot{H}^q} := \|\nabla^q \theta\|_2, \qquad q < 0, \tag{14}$$

which is a negative homogeneous Sobolev norm. This norm can be interpreted for negative q via eigenfunctions of the Laplacian operator. For example, in a periodic domain, we have

$$\|\theta\|^2_{\dot{H}^q} = \sum_{\boldsymbol{k}} |\boldsymbol{k}|^{2q} |\hat{\theta}_{\boldsymbol{k}}|^2, \tag{15}$$

from which we see that, for $q < 0$, $\|\theta\|^q_{\dot{H}}$ smooths θ before taking the L^2 norm. The theorem

$$\lim_{t\to\infty} \|\theta\|_{\dot{H}^q} = 0, \quad q < 0 \quad \iff \quad \theta \text{ converges weakly to } 0, \tag{16}$$

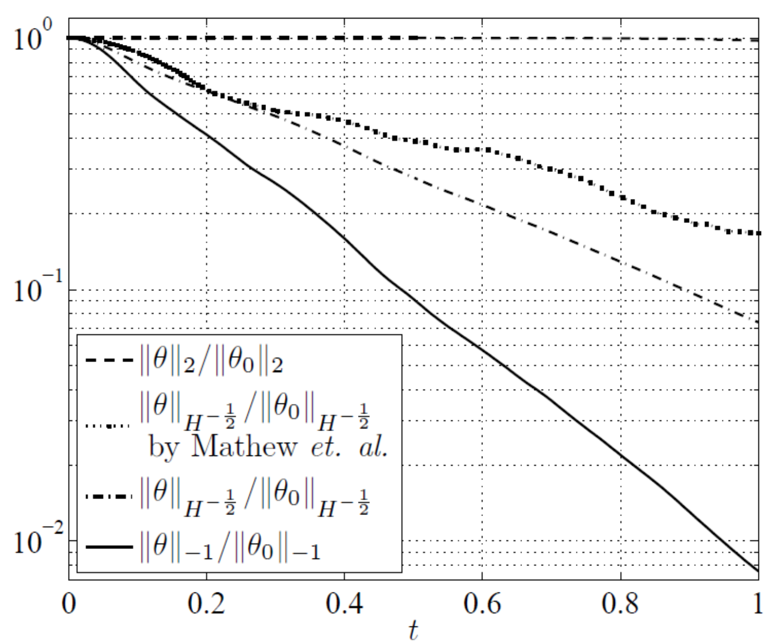

Figure 2: Comparison of the mix-norms for a flow optimized using the separate methods of optimal control and optimal instantaneous decay. Figure from Lin *et al.* [3].

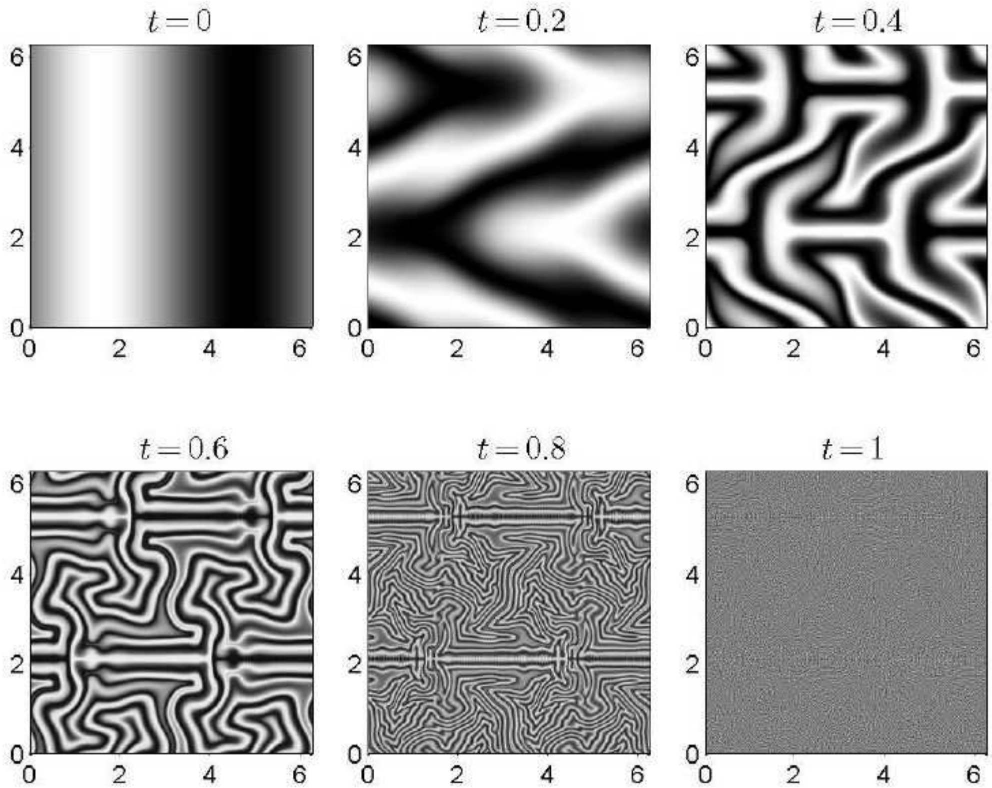

Figure 3: Evolution of the concentration field for the flow optimized in the case $q = -1$ as computed by Lin *et al.* [3].

due to Mathew, Mezic and Petzold [5] and Doering, Lin and Thiffeault [3] shows that we can track any mix-norm to determine whether a system is mixing (in the weak sense). The existence of this quadratic norm facilitates optimization of the velocity field to achieve good mixing. Mathew, Mezic, Grivopoulos, Vaidya and Petzold [4] have used optimal control to optimize the decay of the $q = -1/2$ mix-norm. Lin, Doering and Thiffeault [3] have optimized the instantaneous decay rate of the $q = -1$ norm using the method of steepest descent, which is easier to compute numerically but yields suboptimal, but nevertheless very effective, stirring velocity fields. A comparison of the methods for optimized mixing is shown in figure 2. The solid line decays faster, but this is merely because the $\dot{H}^{-1}$ cannot be compared directly with $\dot{H}^{-1/2}$. The corresponding evolution of the concentration field for the case $q = -1$ from Lin *et al.* [3] is shown in figure 3.

References

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