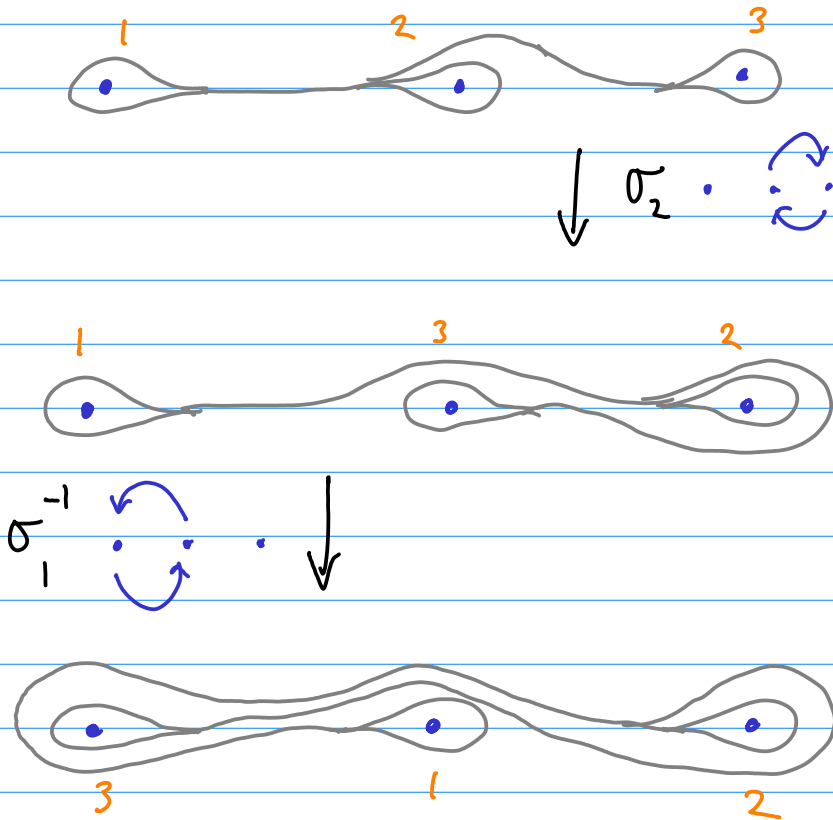# Lecture 25: Computing with train tracks

Train tracks are akin to the foliation being crushed onto a one-dimensional graph. It is a branched manifold.

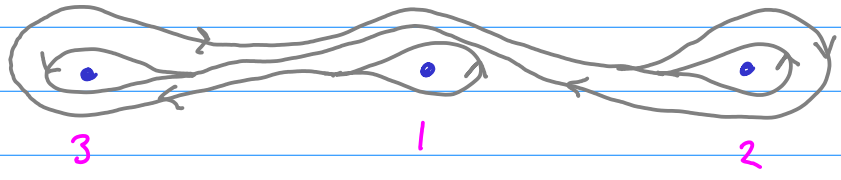The train track must satisfy a similar invariance property to the foliation:

Notice how the final picture "looks" like the initial branched manifold.

We can use this to calculate λ. To do this, we label and orient the edges of the train track.
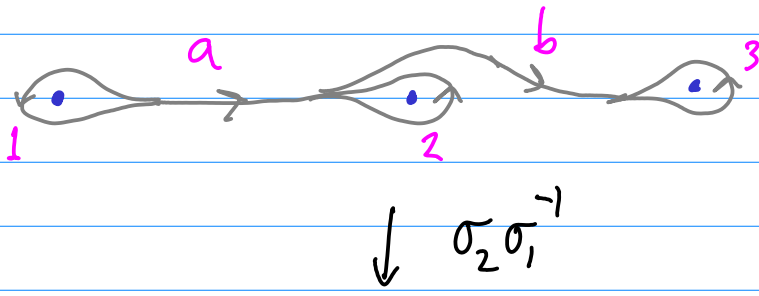
Label "main" edges by letters, and "loops" by numbers.

$$\downarrow \; \sigma_2 \sigma_1^{-1}$$

Now, much like Markov boxes, described the transformed edges in terms of _edge paths_, that is, a sequence of original edges.

Get the _train track map_:

$$
\begin{aligned}
a &\longmapsto \bar{a}\,\bar{1}\,a\,b \\
b &\longmapsto \bar{3}\,\bar{b}\,\bar{a} \\
1 &\longmapsto 2 \\
2 &\longmapsto 3 \\
3 &\longmapsto 1
\end{aligned}
$$

$\bar{a}$ means a^{-1}, with respect to direction.

The loops are simply permuted

If we find we cannot write the images of the edges in terms of original edge paths, we have the wrong train track.

Example:

Cannot be written in terms of

We say that the train track _supports_ the pA.

How to get λ: we Abelianize: only count the occurrences of each edge in the map:

$$\begin{pmatrix} a \\ b \\ 1 \\ 2 \\ 3 \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

The spectral radius of the matrix is the spectral radius of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
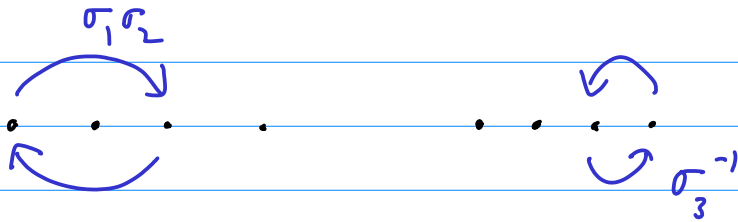
permutation matrix
→ |eigenvalues| = 1

Since loops map only to loops, the <u>transition matrix</u> always has this block triangular form.

More generally, things are of course not so simple. Here are the steps:

1) Find an invariant train track for the diffeo.

2) Verify that the TT map is <u>efficient</u> (no cancellations)

3) Verify that the transition matrix (the sub-block of <u>main edges</u>) is <u>irreducible</u>.

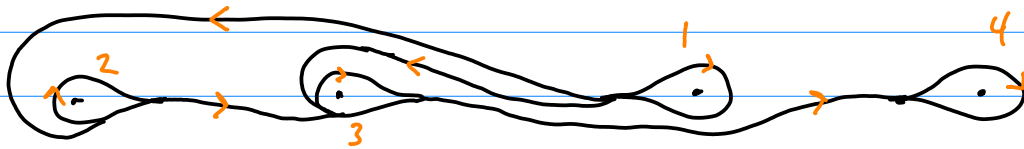We will look at more complicated examples to illustrate these steps.

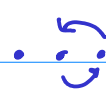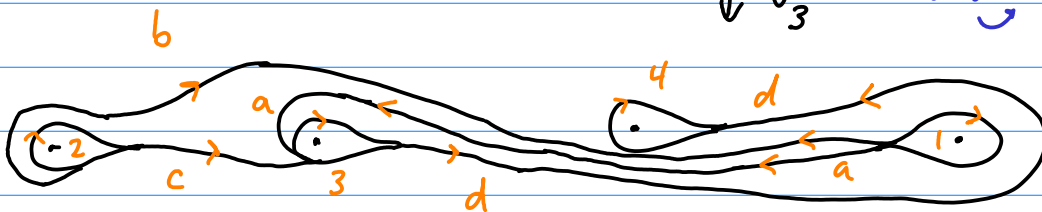Let's do a more complicated example, for the braid $\sigma_1 \sigma_2 \sigma_3^{-1}$:

Try the train track:

three-pronged singularity
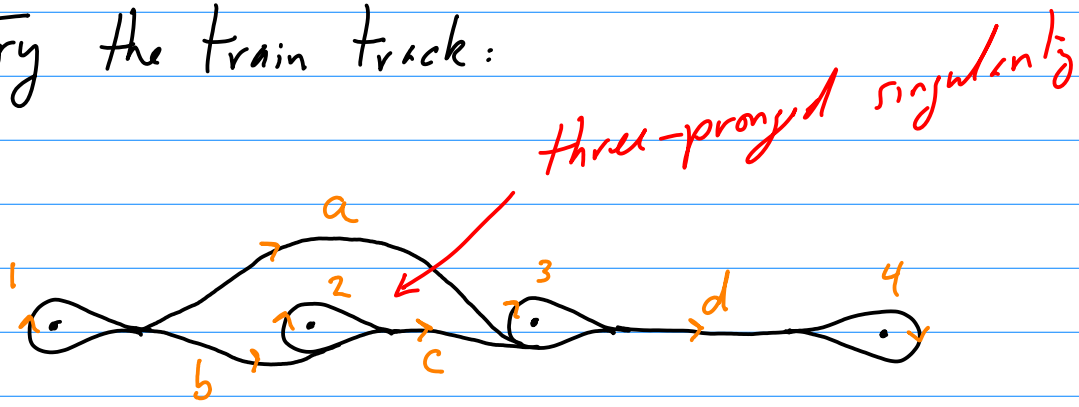
↓ $\sigma_1 \sigma_2$ Skip over the 2 & 3.

↓ σ_3^{-1}

Train track map:

$a \longmapsto \bar{d}\,\bar{c}\,\bar{2}$
$b \longmapsto \bar{d}\,\bar{a}\,\bar{1}$
$c \longmapsto b$
$d \longmapsto c\,d\,\bar{4}\,\bar{d}$
$1 \longmapsto 4$
$2 \longmapsto 1$
$3 \longmapsto 2$
$4 \longmapsto 3$

Incidence matrix:

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{matrix} f(a) \\ f(b) \\ f(c) \\ f(d) \end{matrix}$$

with column labels a b c d.

Can we tell from the incidence matrix if this is a pseudo-Anosov?

We can, as long as the train track map is efficient (later).

$$M^2 = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & 2 & 3 \\ 1 & 2 & 4 & 9 \end{pmatrix}, \quad M^4 = \begin{pmatrix} 1 & 2 & 6 & 12 \\ 2 & 3 & 6 & 14 \\ 0 & 2 & 3 & 6 \\ 2 & 4 & 10 & 21 \end{pmatrix}$$

$$M^5 = \begin{pmatrix} 2 & 6 & 13 & 27 \\ 3 & 6 & 16 & 33 \\ 2 & 3 & 6 & 14 \\ 4 & 10 & 23 & 48 \end{pmatrix}$$

No more zeros!

A matrix A is **reducible** if there exists a permutation matrix P such that $P^T A P$ is block-triangular. An **irreducible** matrix is not reducible. (d'uh)

Equivalently: A is irreducible if, $\forall i, j$, there exists k such that
$$(A^k)_{ij} > 0.$$

Hence, the matrix M above is irreducible.

Perron-Frobenius theorem: Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ij} \geq 0$. Then:

1. The largest eigenvalue of A is the spectral radius, λ. (i.e., the spectral radius is real)

2. The corresponding eigenvector has non negative entries.

3. $\min_i \sum_j a_{ij} \leq \lambda \leq \max_i \sum_j a_{ij}$

[The theorem is slightly different for $a_{ij} > 0$. The the largest eigenvalue is nondegenerate, and the eigenvector has entries > 0.]

3. implies that if we have an irreducible matrix with $\lambda = 1$, then it must be a permutation matrix. (apply 3 to A^n, for n large enough.)

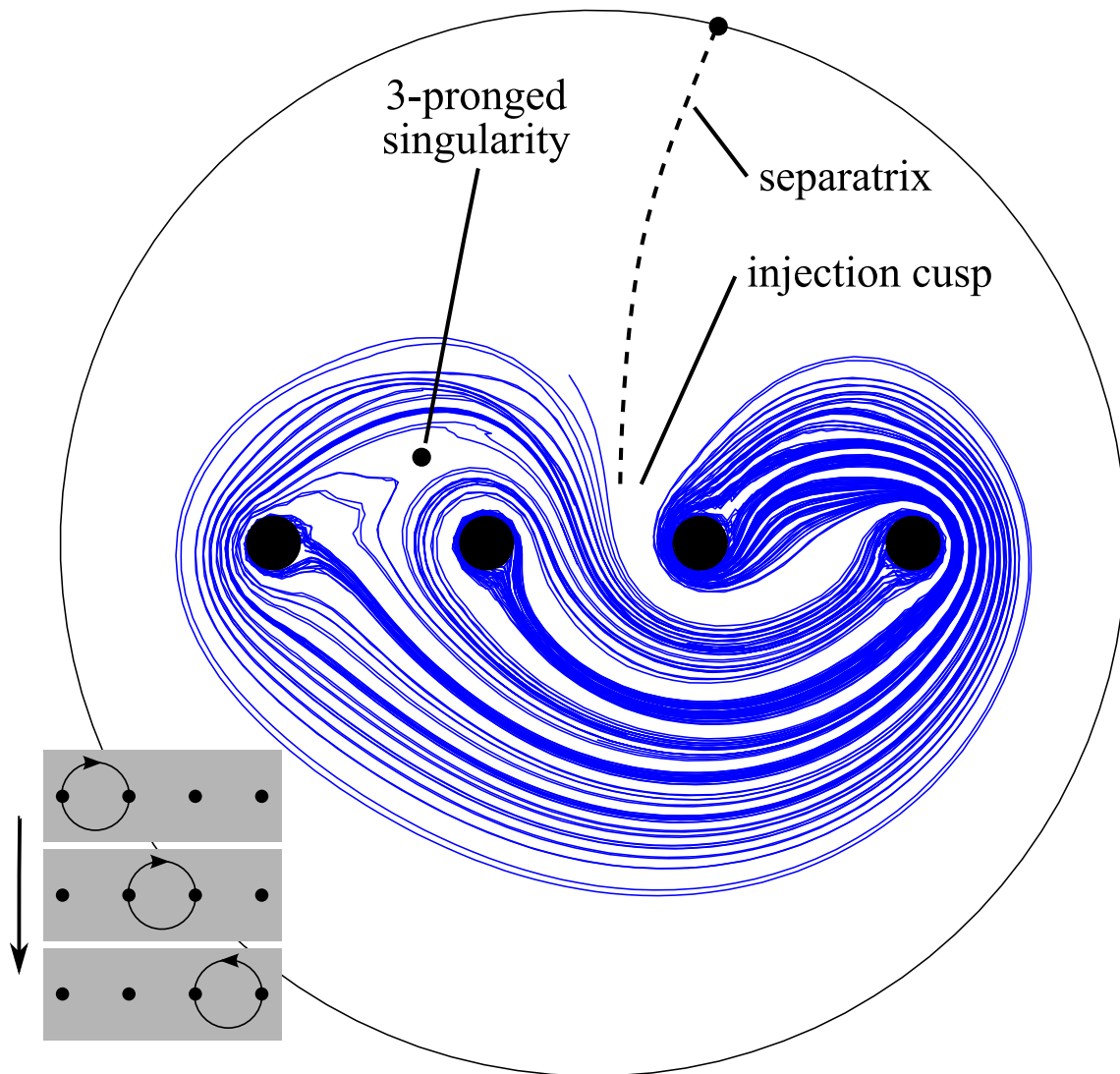For the matrix M above, $\lambda \simeq 2.2966 > 1$, so it is indeed pseudo-Anosov.

Figure 1: A fluid stirred with rod motions corresponding to the braid $\sigma_1\sigma_2\sigma_3^{-1}$ (see J.-L. THIF-FEAULT, M. D. FINN, E. GOUILLART, AND T. HALL, *Topology of chaotic mixing patterns*, Chaos, 18 (2008), p. 033123).