Lecture 24: The Thurston-Nielsen Classification

Our detailed look at the mapping class group of the torus will help us understand the much more general

Thurston-Nielsen Classification Theorem.

Let φ be an orientation-preserving diffeomorphism of an orientable surface S of negative Euler characteristic.

Then φ is isotopic to a diffeomorphism φ' such that either:

1. φ' is finite-order;
2. φ' is reducible;
3. φ' is pseudo-Anosov.

φ' is called the TN representative

We look at each case in turn.

1. Finite-order: $\exists\, m > 0$ s.t. $(\varphi')^m = id$.

We have seen several examples of this for the torus ($|a+d| < 2$ and $[h] = \pm I$.)

Note that, as for the torus, this does not imply that φ itself is finite-order.

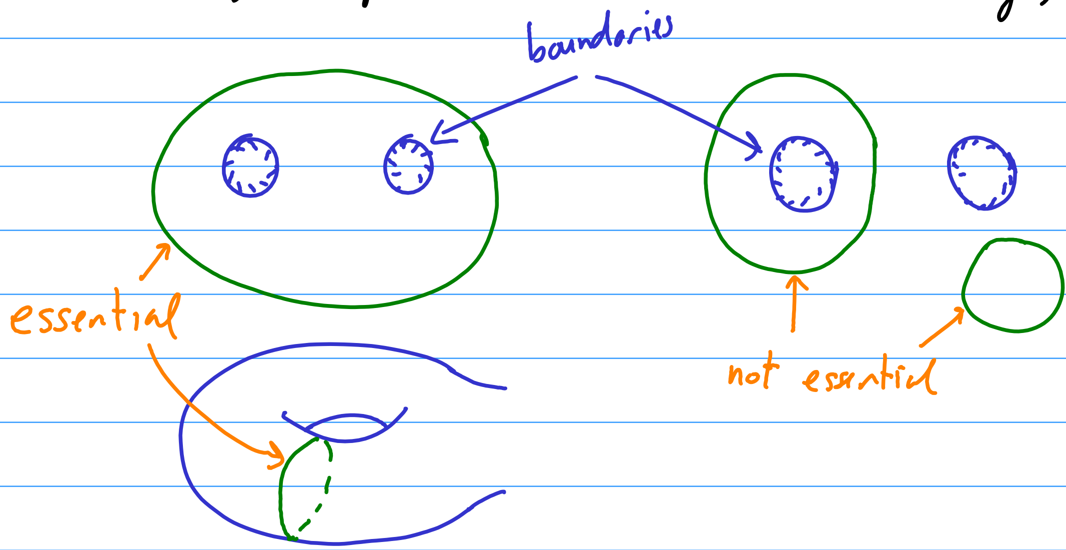
$\Gamma_i \in \mathrm{int}(S)$ Doesn't touch the boundary.

2. Reducible: In this case there is a system Γ of essential simple closed curves $\{\Gamma_1, \dots, \Gamma_n\}$ $(n \geq 1)$ which are pairwise-disjoint with

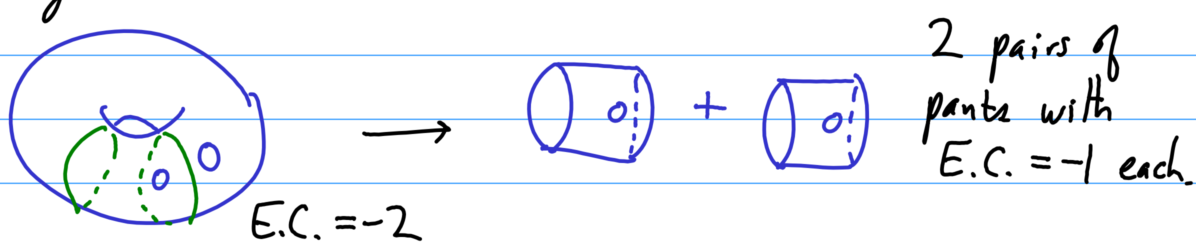
$$\varphi'(\Gamma) = \Gamma. \quad \text{(setwise)}$$

Some clarification: "reducing curves"

- A simple closed curve does not intersect itself
- The Γ_i do not intersect each other (pairwise-disjoint)
- An essential curve Γ_i is not homotopic to a point, to a boundary component, or to each other Γ_j, $j \neq i$.



Another way to capture the "essential" nature of $\Gamma = \{\Gamma_1, \dots, \Gamma_n\}$ is to say that each connected component of $S - \Gamma$ has negative Euler characteristic. (E.C.)



So the E.C. actually place a bound on the number of reducing curves that can exist on a surface.

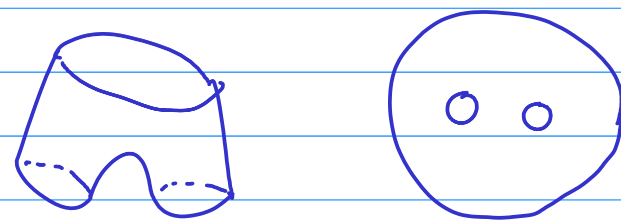
Reminder: Euler characteristic $\chi(S_{g,b,0}) = 2 - 2\underset{\text{genus}}{g} - \underset{\text{\# of boundaries}}{b}$

$\chi(\text{sphere}) = 2$

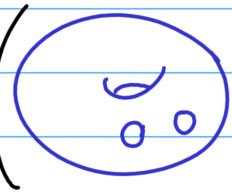
$\chi(\text{disc}) = 1$

$\chi(\text{torus}) = 0$

$\chi(\text{pants}) = -1$



Each extra boundary lowers χ by 1:

$\chi\Big($  $\Big) = 0 - 1 - 1 = -2.$

So the "first" surfaces to which the TN theorem applies are

$S_{2,0,0}$	$S_{1,1,0}$	$S_{0,3,0}$
$\chi = -2$	$\chi = -1$	$\chi = -1$

The other surfaces $(S_{1,0,0}, S_{0,0,0}, S_{0,1,0}, S_{0,2,0})$ have simple MCGs.

The components Γ_i of $\Gamma = \{\Gamma_1, \dots, \Gamma_n\}$ can be permuted among themselves by the action of φ'.

The consequence of φ' being reducible is that we can cut S along Γ, and apply the TN theorem to the individual components. We can repeat this as necessary.

The reducible case "looks" like the case $|a+d| = 2$ for the torus, where the diffeo $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, for example, left an infinity of simple closed curves invariant. However, for the torus we do not have to cut along those curves to continue the classification.

This takes us to the most important case.

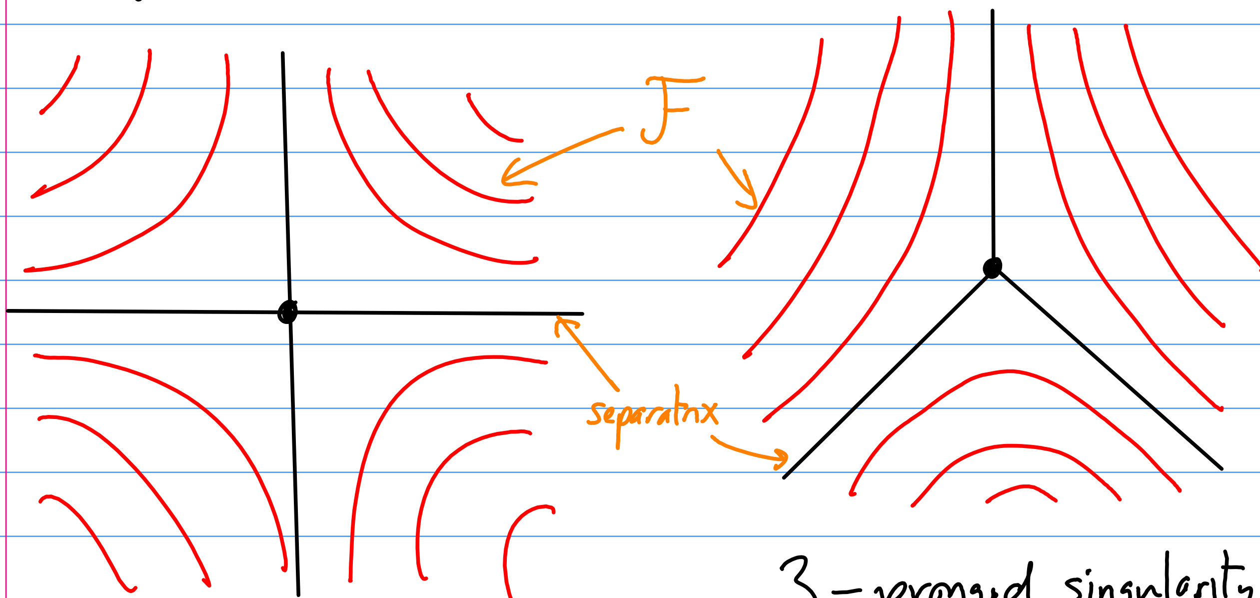
3. pseudo-Anosov (pA)

φ is a pseudo-Anosov diffeomorphism if there exists a real number $\lambda > 1$ (the dilatation or expansion constant) and a pair of transverse measured foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$ such that

$$\varphi(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u)$$
$$\varphi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1} \mu^s)$$

Unlike the torus case, $\mathcal{F}^u$ and $\mathcal{F}^s$ have a finite number of singularities.

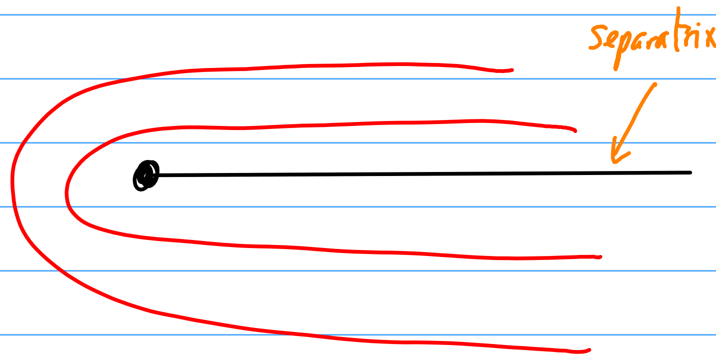
In the neighborhood of a singularity, each $\mathcal{F}$ consists of p separatrices, or prongs:



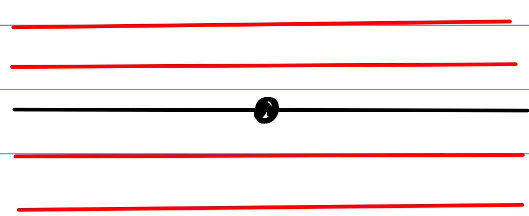
4-pronged singularity

3-pronged singularity

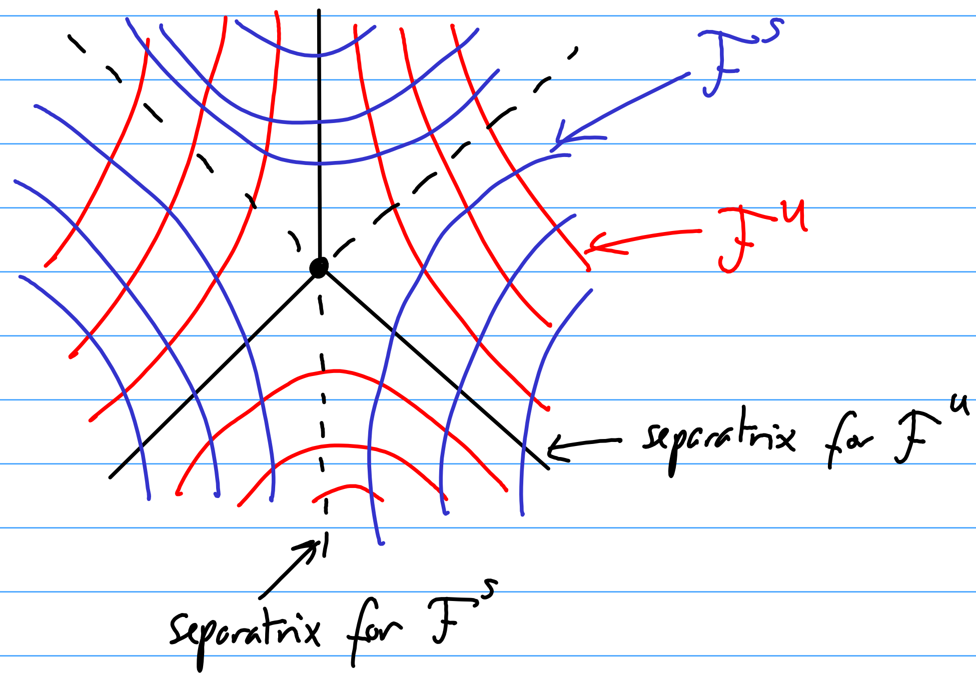
1-pronged singularity



A 2-pronged singularity is special: it is a regular point, so it really isn't a singularity at all:



Near a singularity, $\mathcal{F}^u$ and $\mathcal{F}^s$ remain transverse, except at the singularity point itself:

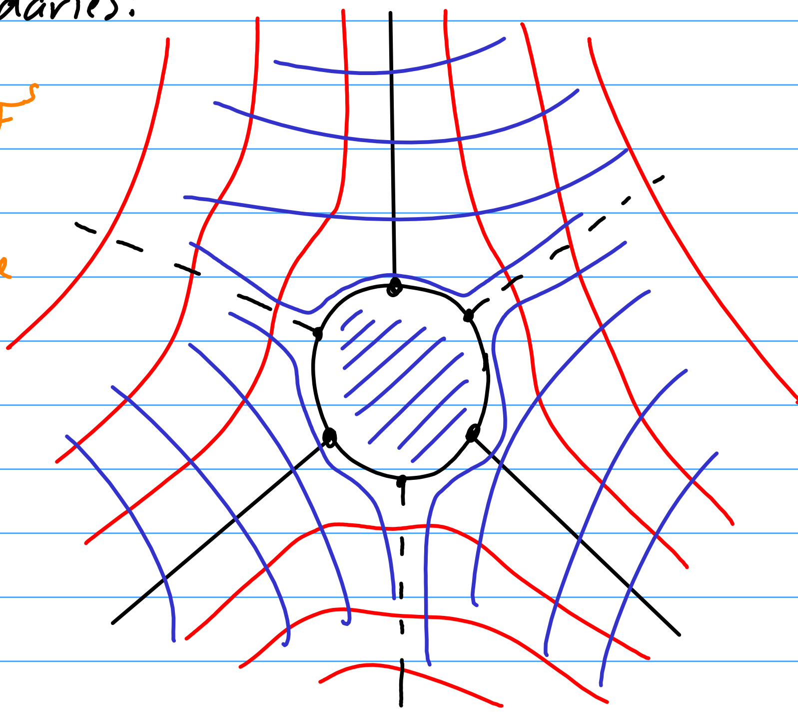


The separatrices are semi-infinite leaves of the foliation.

$\mathcal{F}^u$ and $\mathcal{F}^s$ share singularities and they have the same number of prongs there. (Exception: singularities at boundaries — below).

Near boundaries:

- Singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$ do not coincide on boundaries, though same number of sings.
- Each boundary has at least one sing. for a pA.
- Boundary singularities are 3-prongs.



This is called the "standard model" of pA's (Fried — See Boyland '99)

Boundaries consist of leaves of the foliations, separated by singularities. The leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ are allowed to coincide at the boundaries. This is also the only place where singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$ can have different number of prongs.

The separatrices of $\mathcal{F}^u$ attached to the boundary will hold special significance: