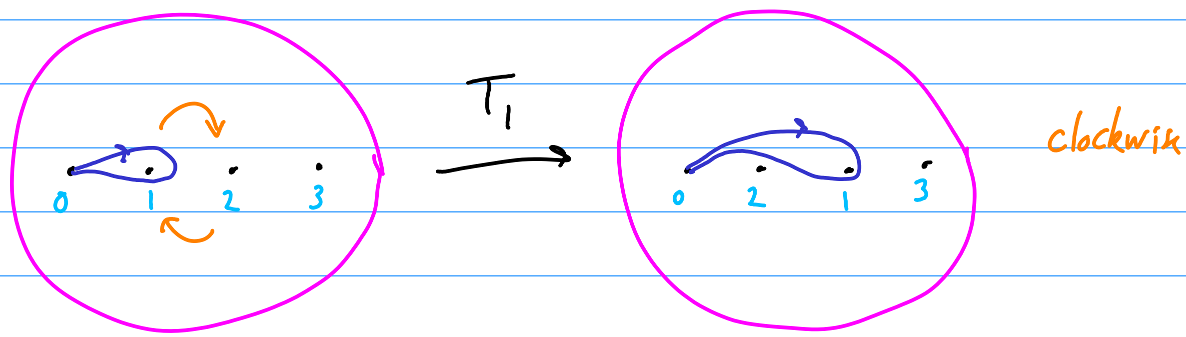
Lecture 23: Topological stirring

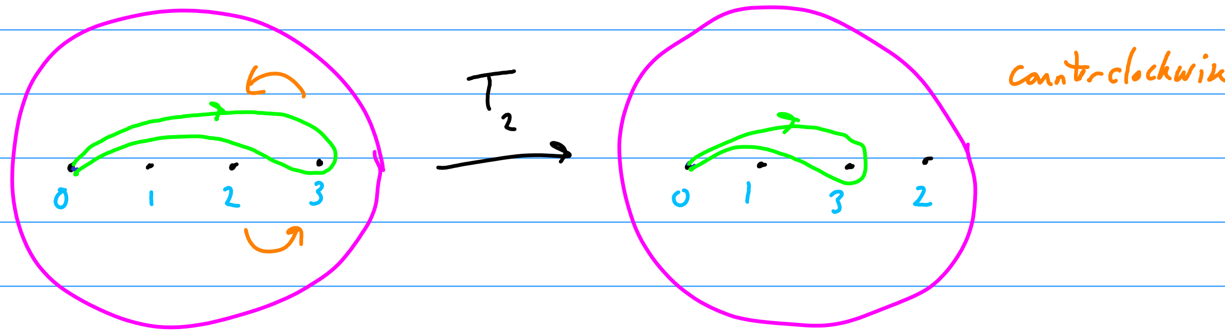
Last time: $T^2/\iota \cong S^2$ $\quad \iota$ = hyperelliptic involution (4 branch points)

Two special mappings: $T_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \bmod 1$ $\qquad T^2 \to T^2$

$T_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \bmod 1$

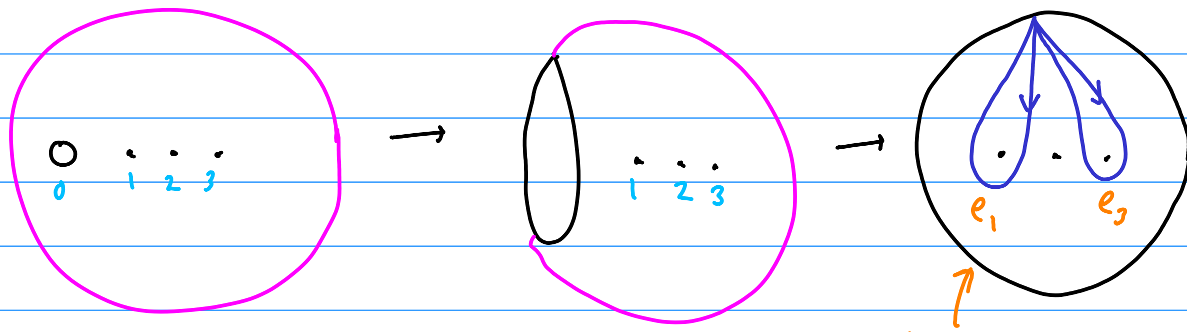
When projected down to the sphere, these can be interpreted as maps that "exchange" punctures $1 \leftrightarrow 2$ or $2 \leftrightarrow 3$.





T_1 and T_2 generate the mapping class group of the sphere with 4 punctures, with one puncture fixed.

Now take out a disk at puncture 0:



boundary of a disk with 3 punctures, not sphere.

Now we can connect homeos on T^2 to motion of points (or <u>rods</u>) in a two-dimensional domain.

$\Rightarrow$ stirring a fluid. [movie]

Movie 1: (boyland 1)

$$T_1 T_2^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Trace < 2, so expect some power $=$ id. (recall classification)

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^4 = \mathrm{id}$$

Movie 2: (boyland 2)

$$T_1^{-1} T_2^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

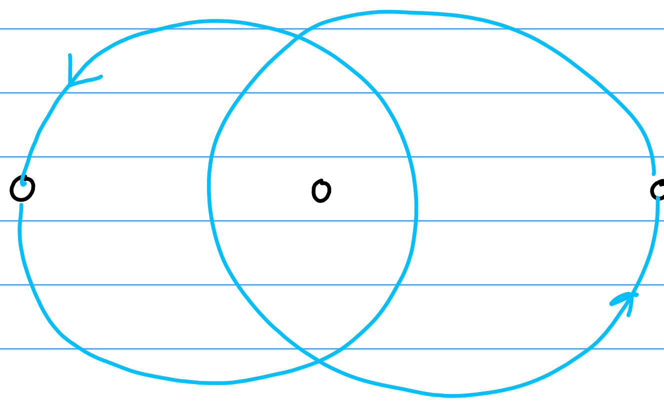
Trace $= 3$ $\Rightarrow$ Anosov on the torus (pseudo-Anosov on disk)

Dilatation $\lambda = \dfrac{\tau + \sqrt{\tau^2 - 4}}{2} = \dfrac{3 + \sqrt{5}}{2} = \varphi^2$

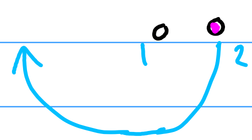
$\varphi = \frac{1}{2}(1 + \sqrt{5})$
= Golden ratio

This tells you something about how "entangled" the fluid motion is.

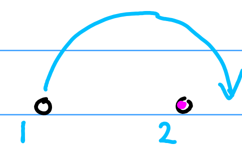
Taffy puller:



Look at movie:

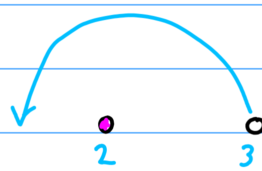


T_1 then

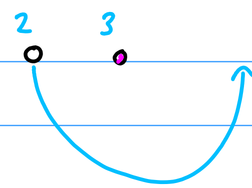


T_1 again

Other side:



T_2 then



T_2 again

Hence, after all the rods return to same initial configuration, we've done

$$T_1^2 T_2^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\lambda = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4}\right) = \frac{1}{2}\left(6 + \sqrt{32}\right)$$

$$\lambda = 3 + 2\sqrt{2} = (1+\sqrt{2})^2$$ Silver ratio!