Lecture 22: From the torus to the sphere

Recall: $f: T^2 \to T^2$ homeomorphism.

Specific map in an isotopy class: $f(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ mod 1

(Note: $f(0,0) = (0,0)$)

$ad - bc = 1$

If $|\text{trace}| > 2$, we have an Anosov homeomorphism.
These are "complex", in the sense that their action on $\pi_1(T^2)$ gives exponential growth in the number of twists.

Now we want to relate this to something more "physical": sphere (eventually disk)

Consider the map: $\iota: T^2 \to T^2$ $\quad \iota\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ mod 1

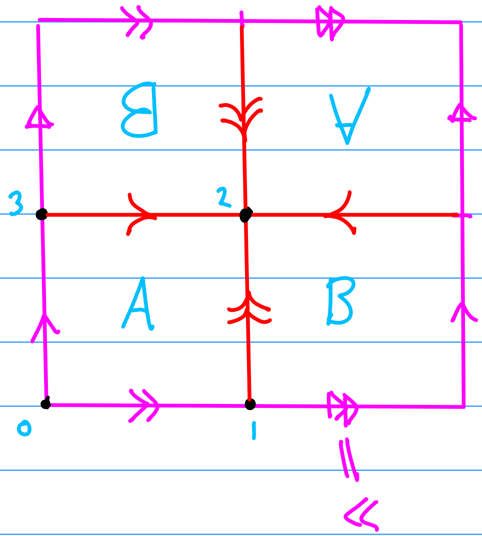
($\iota^2 = id$ involution)

Fixed points: $\begin{matrix} x = -x \text{ mod } 1 \\ y = -y \text{ mod } 1 \end{matrix} \Rightarrow \begin{matrix} x = -x + n \\ y = -y + m \end{matrix} \Rightarrow \begin{matrix} 2x = n \\ 2y = m \end{matrix} \Rightarrow \begin{matrix} x = 0, \frac{1}{2} \\ y = 0, \frac{1}{2} \end{matrix}$

Hence, ι has fixed points $(0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$

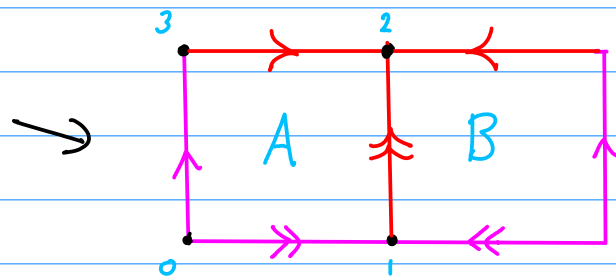
Claim: $T^2/\iota \cong S^2$ (each with 4 points removed)

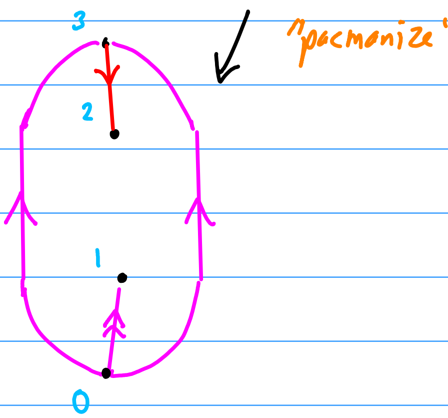
We show this by making the appropriate identifications.

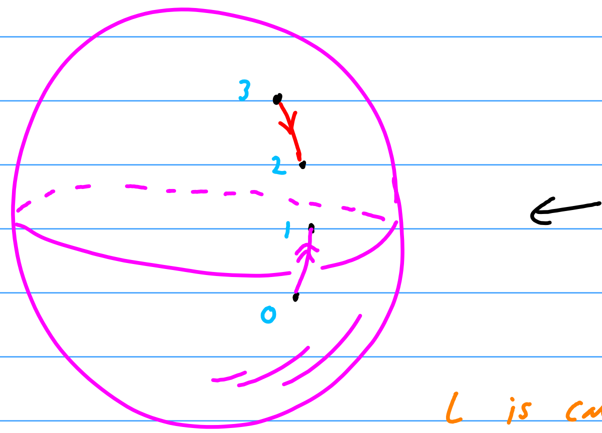


The line $(x, \frac{1}{2}) \to (-x, -\frac{1}{2}) = (1-x, \frac{1}{2})$

The line $(\frac{1}{2}, y) \to (-\frac{1}{2}, -y) = (\frac{1}{2}, 1-y)$





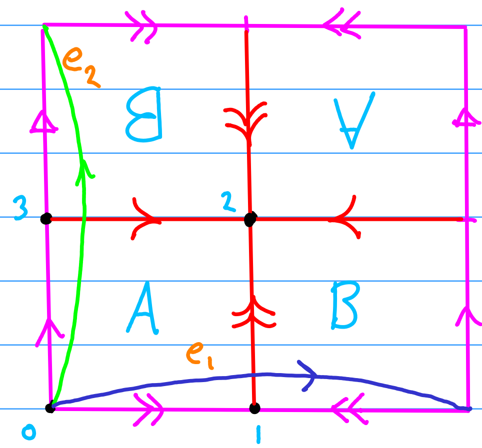


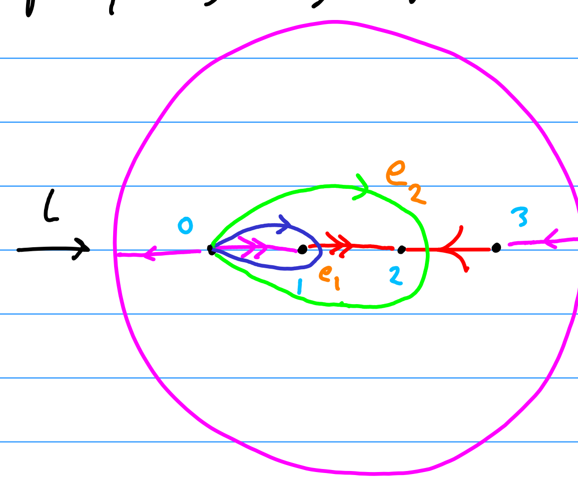
ι is called the hyperelliptic involution

The 4 points are the Weierstrass points.

We say that the torus (with 4 punctures) is a branched double cover of the sphere (with 4 punctures).

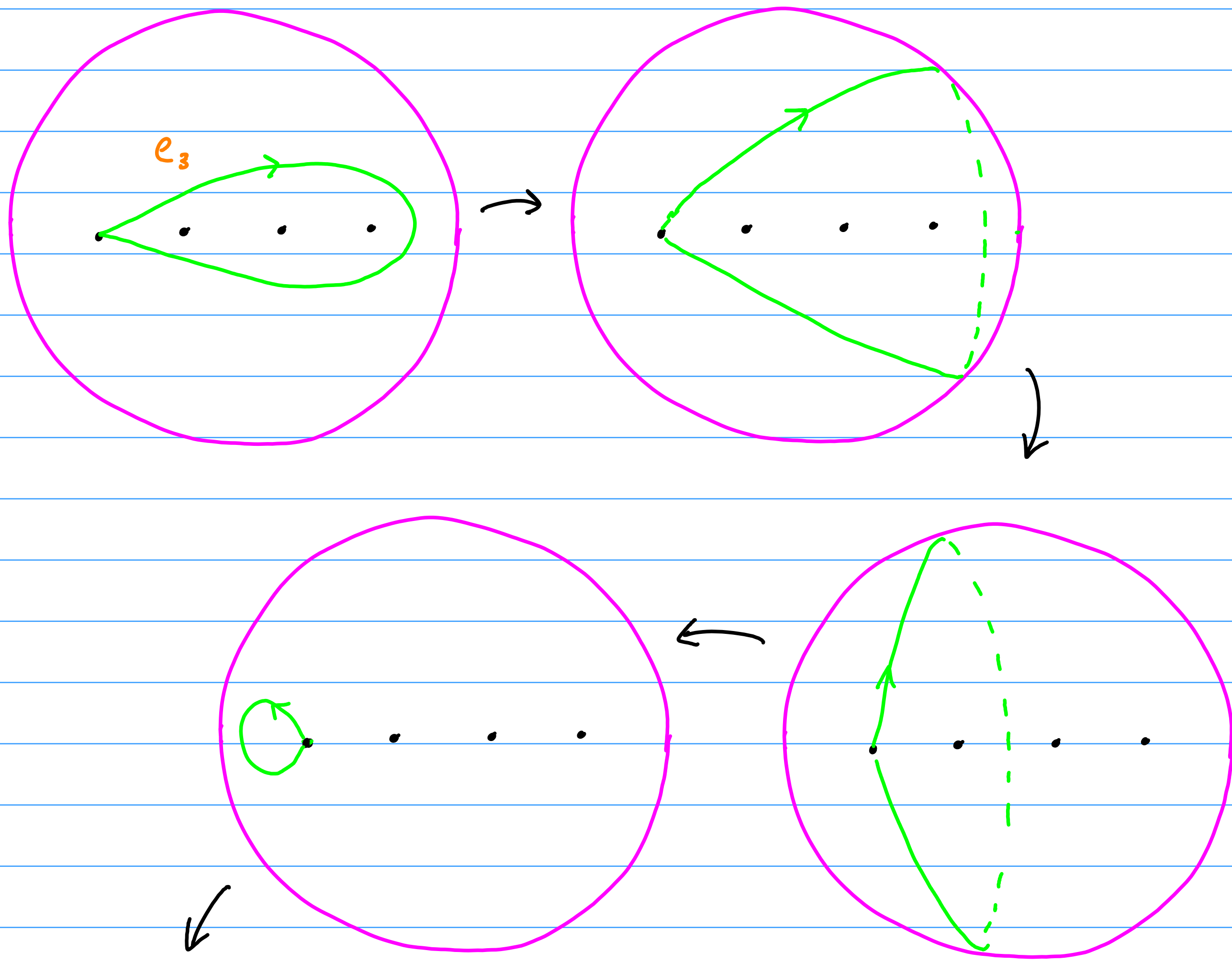
What does an element of $\pi_1(T^2, (0,0))$ map to?

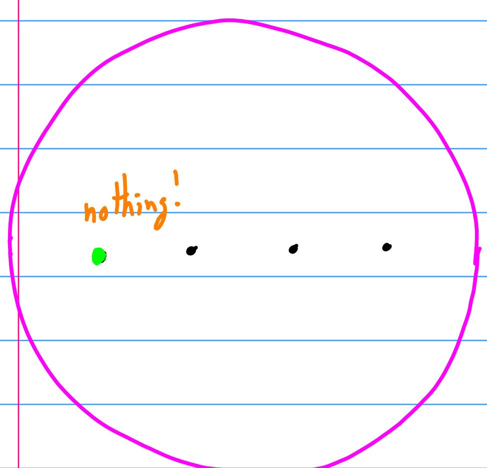




These are loops on the sphere!

Are we missing a loop in $S^2 - 4$ punct., around puncture 3?

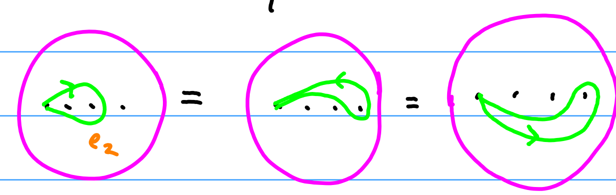




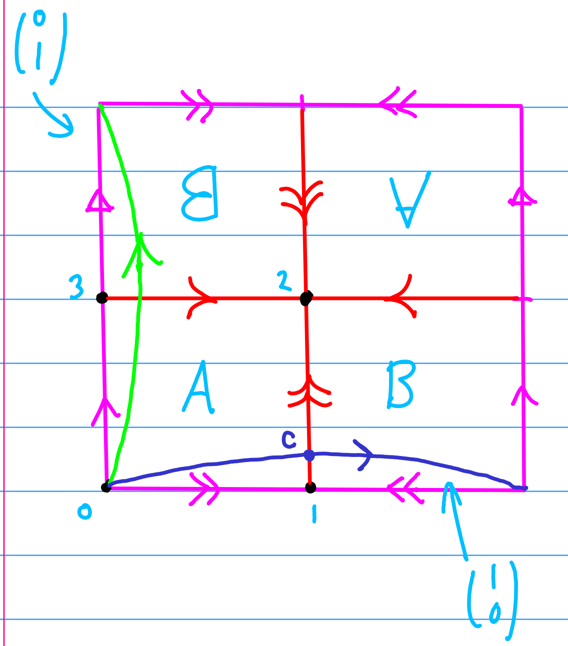
Hence, e_3 = trivial loop

= id in π_1.

Note also:

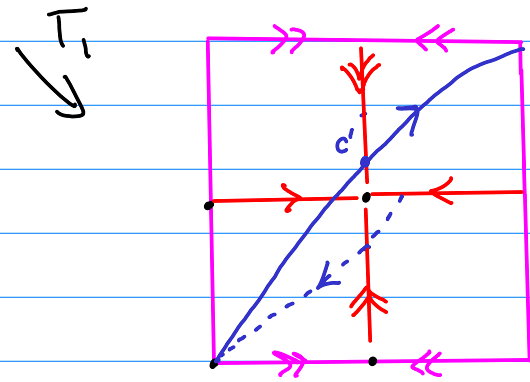


Now consider $T_1(x,y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1$.



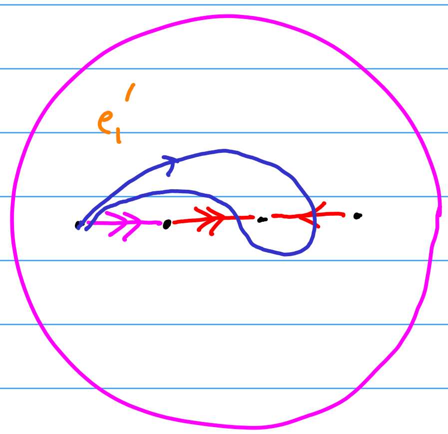
$$T_1\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}, \quad T_1\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$c'=\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1/2\\ \varepsilon\end{pmatrix}=\begin{pmatrix}1/2\\ 1/2+\varepsilon\end{pmatrix}$$

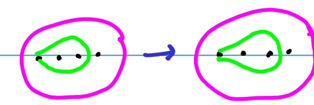


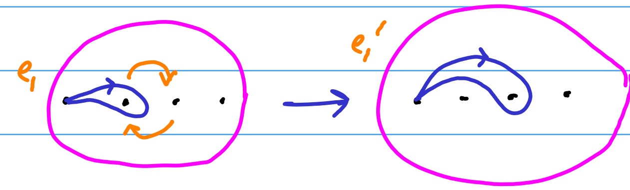
$$T_1\underset{1}{\begin{pmatrix}1/2\\0\end{pmatrix}}=\underset{2}{\begin{pmatrix}1/2\\1/2\end{pmatrix}}, \quad T_1\underset{2}{\begin{pmatrix}1/2\\1/2\end{pmatrix}}=\begin{pmatrix}1/2\\1\end{pmatrix}=\underset{1}{\begin{pmatrix}1/2\\0\end{pmatrix}}, \quad T_1\underset{3}{\begin{pmatrix}0\\1/2\end{pmatrix}}=\begin{pmatrix}0\\1/2\end{pmatrix}=\underset{3}{\begin{pmatrix}0\\1/2\end{pmatrix}}$$

T_1 fixes 0, 3, interchanges 1 and 2.



← image of e_1

(e_2 unchanged: )



Exactly like swapping 1 & 2 clockwise!

Note: Cannot write as e_1, e_2 by dragging, because of the punctures.

T_1^{-1} swaps counterclockwise.

(Can make this clearer by considering homotopy:

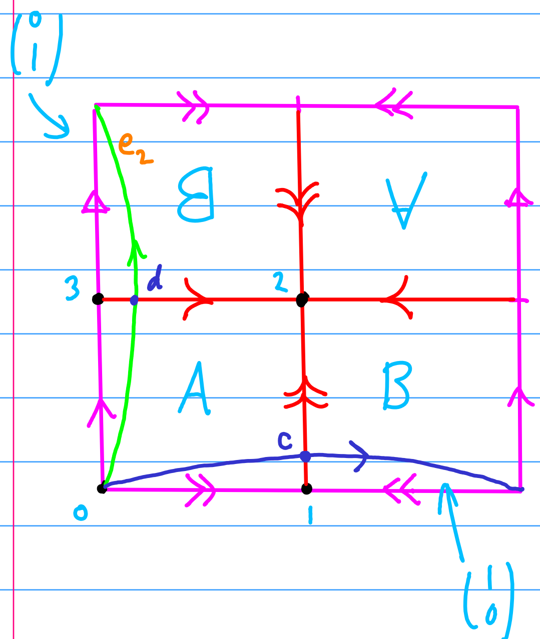
$$T_1(x,y,t)=\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \bmod 1, \quad 0\le t\le 1)$$

Why doesn't it lead to isotopy?
→ not invertible on torus

Also define: $T_2(x,y) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \bmod 1.$

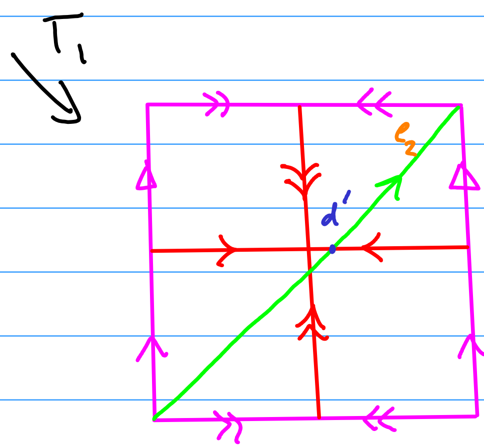
$$\underset{1}{T_2\begin{pmatrix}1/2\\ 0\end{pmatrix}} = \underset{1}{\begin{pmatrix}1/2\\ 0\end{pmatrix}}, \quad \underset{2}{T_2\begin{pmatrix}1/2\\ 1/2\end{pmatrix}} = \begin{pmatrix}1\\ 1/2\end{pmatrix} = \underset{3}{\begin{pmatrix}0\\ 1/2\end{pmatrix}}, \quad \underset{3}{T_2\begin{pmatrix}0\\ 1/2\end{pmatrix}} = \underset{2}{\begin{pmatrix}1/2\\ 1/2\end{pmatrix}}$$

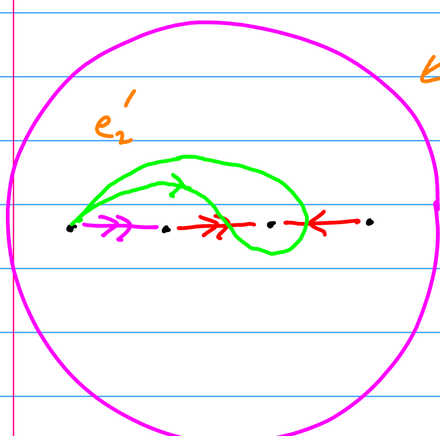
T_2 fixes 0, 1, interchanges 2 and 3.



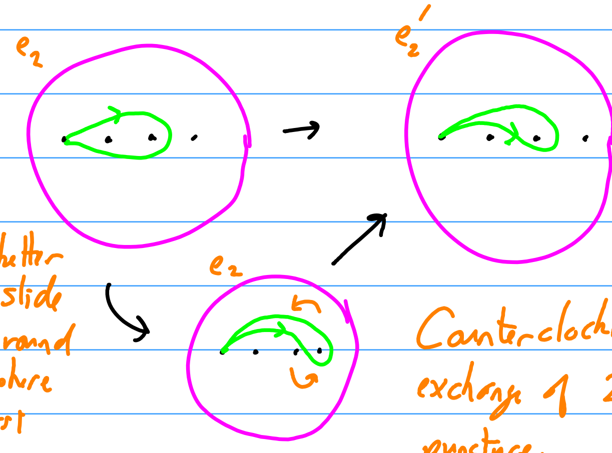
$T_2\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$, $T_1\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$

$d' = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varepsilon\\ 1/2\end{pmatrix} = \begin{pmatrix}\varepsilon\\ 1/2+\varepsilon\end{pmatrix}$



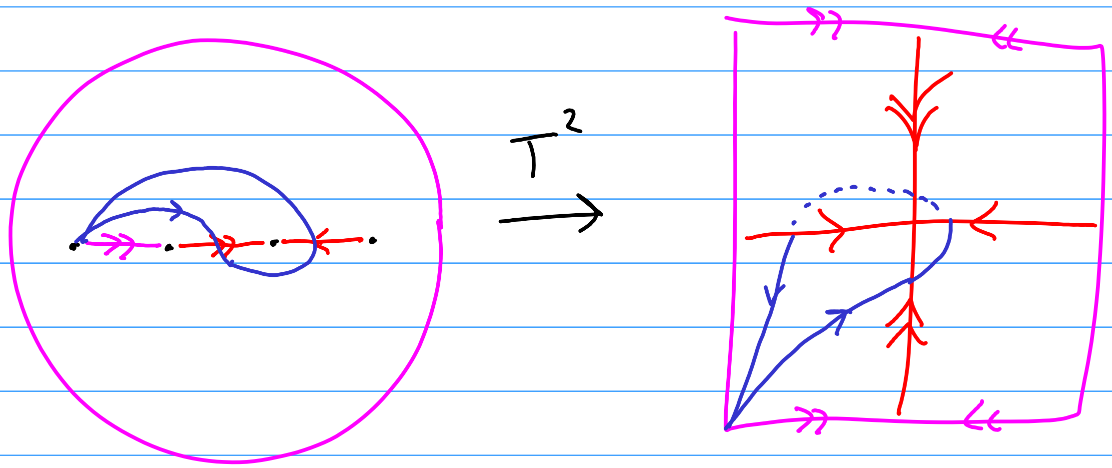


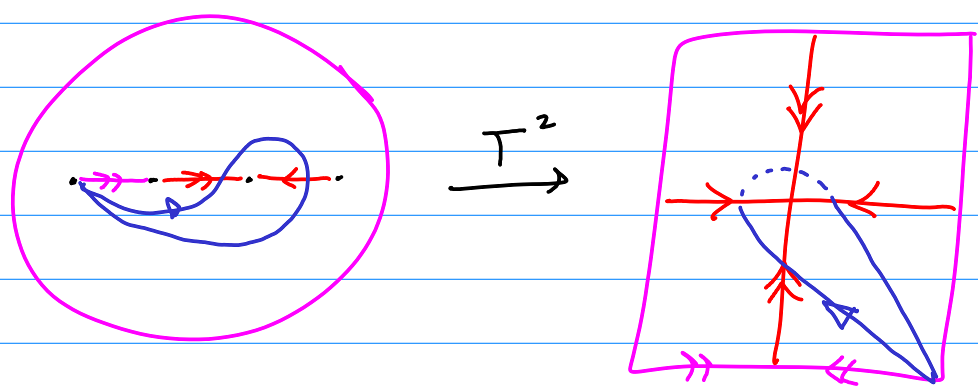
← image of e_2 (e_1 unchanged) under T_2



better slide around sphere first

Counterclockwise exchange of 2nd & 3rd puncture.


T^2


T^2