Lecture 21: Anosov homeomorphisms

Recall $f: T^2 \to T^2$ homeomorphism (orientation-preserving)

$f_*: \pi_1(T^2) \to \pi_1(T^2)$ $\qquad f_* \in SL(2,\mathbb{Z})$

$f_* = M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad ad - bc = 1$

$p(x) = x^2 - \tau x + 1$ characteristic polynomial with $\tau = a + d$
$= \text{trace}$

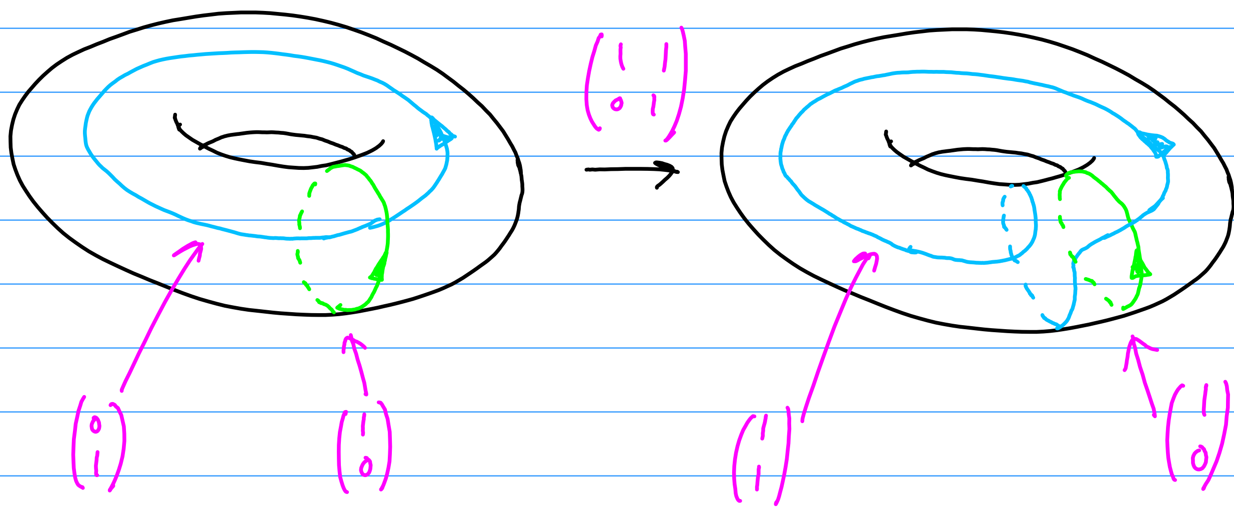
$x = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4})$

So far: 1) $|\tau| < 2 \Rightarrow M^{12} = I$ finite-order

2) $|\tau| = 2 \Rightarrow M = \pm I$ (also finite-order)

or $Mv = \pm v$ for some $v \in \pi_1(T^2)$

example: $\Rightarrow$ invariant loop (reducible)



That is: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ leaves the loop $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ invariant.

Only one possibility left.

3) $|\tau| > 2$: In that case we get two distinct real roots:

$$x_\pm = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2-4}\right), \quad \text{with} \quad x_+ x_- = 1$$

The roots are inverse of each other, and we define

$$\lambda = \max\left(|x_+|, |x_-|\right) > 1.$$

The eigenvectors of M are $\qquad \pm = \mathrm{sign}(\tau)$

$$u = \begin{pmatrix} \pm\lambda - d \\ c \end{pmatrix}, \quad s = \begin{pmatrix} \pm\lambda^{-1} - d \\ c \end{pmatrix}, \quad \lambda > 1$$

Check: (with $\tau > 2$)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda - d \\ c \end{pmatrix} = \begin{pmatrix} a\lambda \overbrace{- ad + bc}^{-1} \\ c\lambda - \cancel{cd} + \cancel{cd} \end{pmatrix} = \begin{pmatrix} \overbrace{(-\lambda^2 + (a+d)\lambda - 1)}^{0} + \lambda^2 - \lambda d \\ c\lambda \end{pmatrix}$$

$$= \lambda \begin{pmatrix} \lambda - d \\ c \end{pmatrix}.$$

Claim: λ is irrational $\leftarrow$ There are easier ways, but this is nice...

Assume $\boxed{\tau > 2}$ (positive): $\leftarrow$ $\tau < -2$ of course has nearly identical proof.

$$\lambda = \tfrac{1}{2}\left(\tau + \sqrt{\tau^2 - 4}\right) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

$$\lambda^2 - \tau\lambda + 1 = 0 \iff \lambda - \tau + \lambda^{-1} = 0$$

$$\lambda = \underbrace{\tau - 1}_{\substack{\text{Positive} \\ \text{integer}}} + \underbrace{(1 - \lambda^{-1})}_{\substack{0 < 1-\lambda^{-1} < 1 \\ \text{Fractional part}}} = a_0 + (1 - \lambda^{-1})$$

$$\therefore \; a_0 = \tau - 1$$

$$\lambda - a_0 = 1 - \lambda^{-1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

$$\frac{1}{1-\lambda^{-1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}$$

ϕ = Golden ratio

$$1 - \lambda^{-1} = 1 - (\tau - \lambda) \quad \text{since} \quad \lambda^2 - \tau\lambda + 1 = 0$$
$$= (1-\tau) + \lambda$$

ϕ^2!

Since $\lambda = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4}\right)$ is monotonic, $\min \lambda = \frac{1}{2}(3+\sqrt{5})$ for $|\tau| = 3$,
$= 2.61803\ldots$

Hence, $1 < \dfrac{1}{1-\lambda^{-1}} \leq \dfrac{1}{1 - \max \lambda^{-1}}$ $\qquad \max \lambda^{-1} = (\min \lambda)^{-1} = \frac{1}{2}(3-\sqrt{5})$

$$= \frac{1}{\frac{1}{2}(\sqrt{5}-1)}$$

$$= \frac{2(\sqrt{5}+1)}{4} = \frac{1}{2}(1+\sqrt{5}) < 2.$$

$\therefore 1 < \dfrac{1}{1-\lambda^{-1}} < 2.$ Conclude: $a_1 = 1.$

$$\left(\frac{1}{1-\lambda^{-1}} - 1\right)^{-1} = a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}$$

$$\left(\frac{1}{1-\lambda^{-1}} - 1\right)^{-1} = \left(\frac{\cancel{1} - (\cancel{1} - \lambda^{-1})}{1-\lambda^{-1}}\right)^{-1} = \left(\frac{1}{\lambda - 1}\right)^{-1} = \lambda - 1.$$

$$\lambda - 1 = a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}} \iff \lambda = \underbrace{(a_2+1)}_{a_0 = \tau - 1} + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}$$

Same expression as when we started!

$\therefore a_2 = \tau - 2.$

One more time:

$$\lambda^2 - \tau\lambda + 1 = 0$$
$$\lambda - \tau + \lambda^{-1} = 0$$
$$\lambda - \tau + 2 = 2 - \lambda^{-1}$$

$$(\lambda - 1 - a_2)^{-1} = a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ldots}}$$

$$(\lambda - 1 - a_2)^{-1} = (\lambda - \tau + 1)^{-1} = (1 - \lambda^{-1})^{-1}$$

$$\frac{1}{1 - \lambda^{-1}} = a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ldots}} \qquad \therefore \quad \begin{aligned} a_3 &= a_1 \\ a_4 &= a_2 \\ a_5 &= a_1 \\ a_6 &= a_2 \ \ldots \end{aligned} \quad \text{Periodic!}$$

We've seen this before!

Continued fraction representation:

$$\lambda = [\tau - 1;\ \underbrace{1,\ \tau - 2}_{\text{period-2}},\ 1,\ \tau - 2,\ \ldots]$$

Of course, this shouldn't have surprised us: the irrational solutions of quadratic equations with integer coefficients are periodic. But this polynomial has a particularly simple form.

More importantly, this shows that λ is irrational for any $|\tau| > 2$, (negative τ proceeds identically)

$[f]$ is the isotopy class of an Anosov homeomorphism with dilatation $\lambda > 1$.