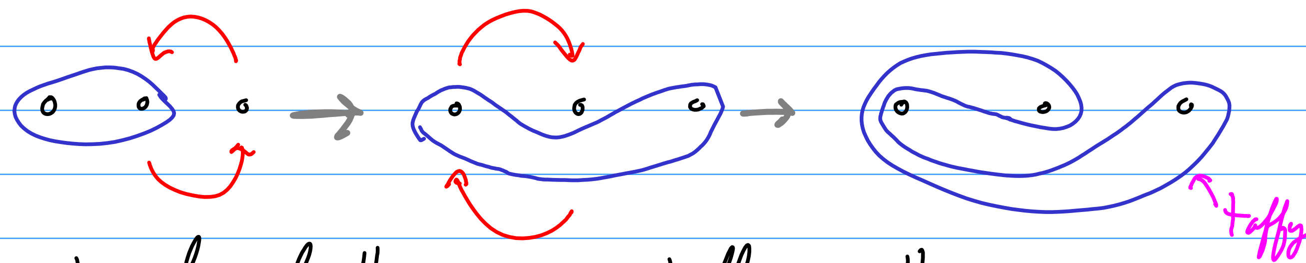
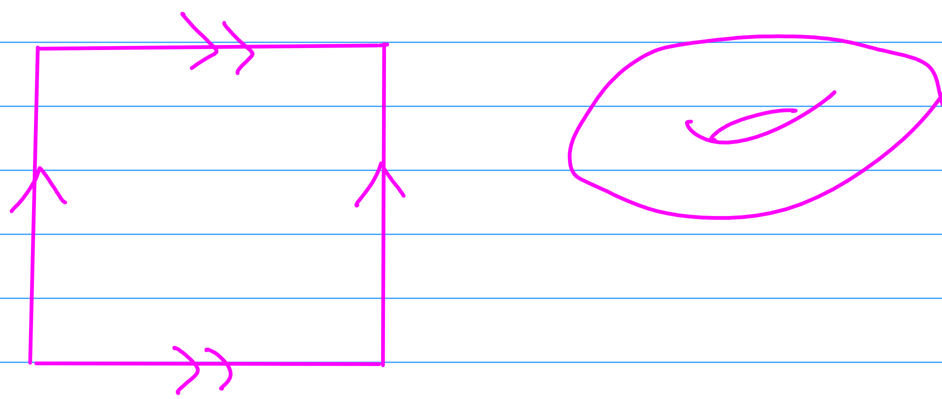
Lecture 20: Topological mixing on the torus

Stirring by moving rods [movie] $\begin{cases} \text{fluids (viscous)} \\ \text{elastic bodies (bread, taffy)} \end{cases}$



Repeat: line length grows exponentially in this case.

How do we characterize this? A lot of insight obtained from first considering the torus.



T^2

Homeomorphisms $T^2 \to T^2$ Invertible, continuous with continuous inverse.

← orientation-preserving

$\mathrm{Homeo}_0^+(T^2)$.

$\mathrm{Homeo}^+(T^2)$ is a group under composition of functions.

Define:

$$MCG(T^2) = Homeo^+(T^2) / \text{isotopy}$$

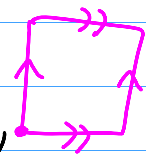
Mapping Class Group of T^2

(inherits the group structure of $Homeo^+(T^2)$)

assume they have a fixed point $Homeo^+(T^2, x_0)$

What does $MCG(T^2)$ look like?

Consider an induced homomorphism on $\pi_1(T^2, x_0)$



↳ (0,0)

fundamental group of T^2, with loops based at x_0

$$f: T^2 \to T^2, \qquad f_*: \underbrace{\pi_1(T^2)}_{\mathbb{Z}^2} \to \pi_1(T^2)$$

linear

$f_*([f]+[g]) = f_*[f]$
$f_*[g]$

Hence, f_* given by matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a,b,c,d \in \mathbb{Z}$$

But also $f \circ f^{-1} = id \Rightarrow f_* \circ f_*^{-1} = I$ so $\overbrace{f_*}^{\in GL(2,\mathbb{Z})}$ invertible

$ad - bc \neq 0$

Let $m = ad - bc \neq 0$. We have also $f_*^{-1}: \mathbb{Z}^2 \to \mathbb{Z}^2$, so

$$f_*^{-1}: \quad \frac{1}{m}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

, so need all entries $\in \mathbb{Z}$.

↳ m divides every entry

Let $a = m\alpha$, $b = m\beta$, $c = m\gamma$, $d = m\delta$, $\alpha, \beta, \gamma, \delta$ integers.

Then $m = ad - bc = m^2(\alpha\delta - \beta\gamma) \Rightarrow 1 = m(\alpha\delta - \beta\gamma)$.

Since all integers, need $m = \pm 1$. $m = +1 \Rightarrow$ orientable.

Hence, $\boxed{MCG(T^2) = SL(2,\mathbb{Z})}$ Why is this?

Now, how do we classify the elements of this group?

Look at eigenvalues. $\det\left[\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M\ (=f_*)} - xI\right] = x^2 - (a+d)x + \underbrace{ad-bc}_{1}$

Let $\tau = a+d$ (trace) note: $p(M) = M^2 - \tau M + I = 0$ Cayley-Hamilton thm

Characteristic polynomial: $p(x) = x^2 - \tau x + 1$

Eigenvalues: $x = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4})$ So $|\tau| = 2$ important

Let's examine different cases.

1) $|\tau| < 2$. $\tau = -1, 0, 1$.

If $\tau = 0$, then $p(M) = M^2 + I = 0 \Rightarrow M^2 = -I \Rightarrow \boxed{M^4 = I}$

If $\tau = \pm 1$, $p(M) = M^2 \mp M + I \Rightarrow M^2 = \pm M - I$

$$M^3 = M(\pm M - I)$$
$$= \pm M^2 - M$$
$$= \pm(\pm M - I) - M$$
$$= \mp I \qquad \boxed{M^6 = I}$$

Either way, we can write

$$\boxed{M^{12} = I, \quad |\tau| < 2}$$

This is called finite-order. After applying f enough times, it is isotopic to the identity map.

2) $|\tau| = 2$: Then eigenvalues are both ± 1 $(= \tau/2)$

$$M^2 \mp 2M + I = (M \mp I)^2 = 0 \Rightarrow M = \pm I + N, \quad N^2 = 0$$

$\begin{pmatrix} a - \tau/2 \\ c \end{pmatrix}$ is the eigenvector: $\leftarrow$ This is for $c \neq 0$ otherwise take $\begin{pmatrix} b \\ d - \tau/2 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a - \tau/2 \\ c \end{pmatrix} = \begin{pmatrix} a(a - \frac{\tau}{2}) + bc \\ c(a - \frac{\tau}{2}) + cd \end{pmatrix} = \begin{pmatrix} a(a - \frac{\tau}{2}) + \overbrace{(bc - ad)}^{-1} + ad \\ c(\underbrace{a + d}_{\tau} - \tau/2) \end{pmatrix}$$

$$= \begin{pmatrix} a(\tau/2) - 1 \\ c\tau/2 \end{pmatrix} = \frac{\tau}{2}\begin{pmatrix} a - \tau/2 \\ c \end{pmatrix} \quad \text{since } 2/\tau = \tau/2$$

$$(\tau/2 = \pm 1)$$

Hence, the homotopy classes given by $\begin{pmatrix} a - \tau/2 \\ c \end{pmatrix}$ are invariant (or reverse direction) under M.

$\Rightarrow$ invariant curve (called reducible)

Let $R = \begin{pmatrix} 1 & a-\tau/2 \\ 0 & c \end{pmatrix}$. Then:

$$R^{-1} M R = \begin{pmatrix} \tau/2 & 0 \\ 1 & \tau/2 \end{pmatrix}$$ Jordan form

other loops not invariant

Simplest type: $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Next time: case 3) $|\tau| > 2$!