

Lecture 19: Biomixing, part 4: Viscous swimmer

1 The dumbbell swimmer

The Green's function for the Stokes equation

$$-\nabla p + \mu \nabla^2 \mathbf{u} = -\mathbf{f} \delta(\mathbf{r}), \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

is $\mathbf{f} \cdot \mathbb{G}(\mathbf{r})$, where $\mathbb{G}(\mathbf{r})$ is the Oseen tensor:

$$\mathbb{G}(\mathbf{r}) = \frac{1}{8\pi\mu\|\mathbf{r}\|} \left(\mathbb{I} + \frac{\mathbf{r}\mathbf{r}}{\|\mathbf{r}\|^2} \right). \quad (2)$$

We model the dumbbell swimmer as two Stokeslets along the z axis Hernandez-Ortiz *et al.* (2005):

$$\mathbf{u}(\mathbf{r}) = F \hat{\mathbf{z}} \cdot \mathbb{G}(\mathbf{r} - A \hat{\mathbf{z}}) + f \hat{\mathbf{z}} \cdot \mathbb{G}(\mathbf{r} - a \hat{\mathbf{z}}). \quad (3)$$

Force balance then dictates

$$F + f = 0, \quad (4)$$

so that

$$\mathbf{u}(\mathbf{r}) = F \hat{\mathbf{z}} \cdot (\mathbb{G}(\mathbf{r} - A \hat{\mathbf{z}}) - \mathbb{G}(\mathbf{r} - a \hat{\mathbf{z}})). \quad (5)$$

Setting $A = 0$ momentarily, note that

$$\lim_{a \rightarrow 0} \frac{1}{a} (\mathbb{G}(\mathbf{r}) - \mathbb{G}(\mathbf{r} - a \hat{\mathbf{z}})) = \hat{\mathbf{z}} \cdot \nabla \mathbb{G}(\mathbf{r}). \quad (6)$$

Recall that $\nabla \mathbf{r} = \mathbb{I}$, $\nabla \|\mathbf{r}\| = \hat{\mathbf{r}}$; we have

$$\begin{aligned} 8\pi\mu \partial_i \mathbb{G}_{jk}(\mathbf{r}) &= \partial_i \left(\frac{1}{\|\mathbf{r}\|} \right) \left(\delta_{jk} + \frac{r_j r_k}{\|\mathbf{r}\|^2} \right) + \frac{1}{\|\mathbf{r}\|} \partial_i \left(\frac{r_j r_k}{\|\mathbf{r}\|^2} \right) \\ &= -\frac{r_i}{\|\mathbf{r}\|^3} \left(\delta_{jk} + \frac{r_j r_k}{\|\mathbf{r}\|^2} \right) + \frac{1}{\|\mathbf{r}\|^3} (\delta_{ij} r_k + r_j \delta_{ik}) - 2r_i \frac{r_j r_k}{\|\mathbf{r}\|^5} \\ &= \frac{1}{\|\mathbf{r}\|^3} \left(\delta_{ij} - 3 \frac{r_i r_j}{\|\mathbf{r}\|^2} \right) r_k + \frac{1}{\|\mathbf{r}\|^3} (\delta_{ik} r_j - \delta_{jk} r_i). \end{aligned}$$

The first term (symmetric in i and j) is the *stresslet*:

$$\mathbb{S}_{ijk} := \frac{1}{8\pi\mu\|\mathbf{r}\|^3} \left(\delta_{ij} - 3\frac{r_i r_j}{\|\mathbf{r}\|^2} \right) r_k. \quad (7)$$

The second term (antisymmetric in i and j) is the *rotlet*:

$$\mathbb{R}_{ijk} := \frac{1}{8\pi\mu\|\mathbf{r}\|^3} (\delta_{ik}r_j - \delta_{jk}r_i). \quad (8)$$

Hence,

$$\begin{aligned} \mathbf{u}(\mathbf{r}) &\sim aF \hat{\mathbf{z}}\hat{\mathbf{z}} : \nabla\mathbb{G}(\mathbf{r}) \\ &= \frac{aF}{8\pi\mu\|\mathbf{r}\|^2} \left\{ \left(1 - 3\frac{zz}{\|\mathbf{r}\|^2} \right) \frac{\mathbf{r}}{\|\mathbf{r}\|} + \mathcal{O}(a^2/\|\mathbf{r}\|^2) \right\} \end{aligned} \quad (9)$$

is the far-field form of the dipole—a pure stresslet. We simply replace a by $(a - A)$ to restore $A \neq 0$, since the corrections incurred are of higher order.

Now assume our dumbbell swimmer is in a frame moving at constant velocity $U\hat{\mathbf{z}}$, so there is an apparent flow $-U\hat{\mathbf{z}}$. We take the positions $a(t)$ and $A(t)$ to be time-periodic in the comoving frame. The forces exerted on the fluid are due to drag on a sphere of radius R at $x = A(t)$ and a sphere of radius $r(t)$ at $x = a(t)$:

$$F(t) = 6\pi\mu R(U + \dot{A}(t)), \quad f(t) = 6\pi\mu r(t)(U + \dot{a}(t)). \quad (10)$$

We take the frame to move at the mean swimming velocity U ; this is obtained from the constraint that the time-averaged velocities of the Stokeslets must vanish in the comoving frame:

$$\langle \dot{A} \rangle = \langle \dot{a} \rangle = 0. \quad (11)$$

From (10) and (4), we have

$$R(U + \dot{A}(t)) = -r(t)(U + \dot{a}(t)), \quad (12)$$

which upon time-averaging gives

$$RU = -U\langle r \rangle - \langle r\dot{a} \rangle, \quad (13)$$

and so the mean swimming velocity is

$$U = -\langle r\dot{a} \rangle / (R + \langle r \rangle). \quad (14)$$

The prescribed functions are $\dot{a}(t)$ and $r(t)$; U is then obtained from (14) and $\dot{A}(t)$ from (12).

The simplest time-dependence we can put is

$$a(t) = a_0 + a_1 \cos(2\pi t/\tau), \quad r(t) = r_0 + r_1 \sin(2\pi t/\tau + \phi) \quad (15)$$

which gives

$$U = \frac{\pi a_1 r_1 \cos \phi}{\tau(r_0 + R)}. \quad (16)$$

The phase $\phi = 0$ yields the fastest mean swimming velocity: it corresponds to the sphere expanding during the power stroke, and shrinking during the recovery stroke. We thus set $\phi = 0$ for simplicity. We then have

$$\dot{A}(t) = -U - r(t)(U + \dot{a}(t))/R \quad (17)$$

$$= \frac{\pi a_1 r_1}{R\tau} \left\{ \left(\frac{2r_0}{r_1} - \frac{r_1}{r_0 + R} \right) \sin(2\pi t/\tau) - \cos(4\pi t/\tau) \right\} \quad (18)$$

which can be integrated to find $A(t) = \int^t \dot{A} dt$. We choose the integration constant to be zero, so that the swimmer's main body oscillates about the origin in the comoving frame.

The far-field stresslet coefficient from (9) with $a \rightarrow (a - A)$ is

$$\frac{(a - A)F}{8\pi\mu} = -\frac{3}{4}(a - A)r(U + \dot{a}), \quad (19)$$

which is a complicated function involving many harmonics. The time-averaged coefficient of the stresslet has the simple form

$$\frac{1}{8\pi\mu} \langle (a - A)F \rangle = \frac{3\pi a_0 a_1 r_1}{4\tau(1 + r_0/R)} = \frac{3}{4} a_0 R U. \quad (20)$$

2 Particle displacement

We now address the question of particle displacements due to a moving stresslet, when the stresslet is aligned with the direction of motion. The other case (when the stresslet is perpendicular to the direction of motion) is more complicated, since it is no longer axially symmetric.

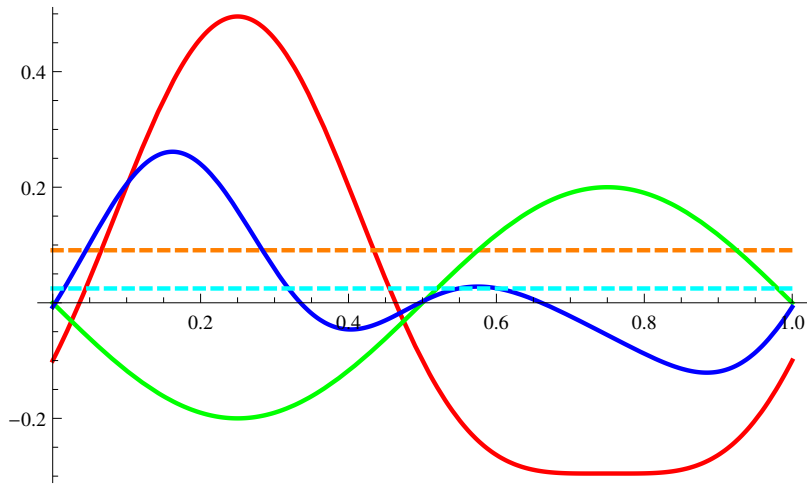


Figure 1: Body velocity \dot{A} (red), flagellum velocity \dot{a} (green), and stresslet coefficient (blue) as a function of time. The dashed lines are the time-averaged stresslet coefficient (cyan) and swimming velocity (orange).

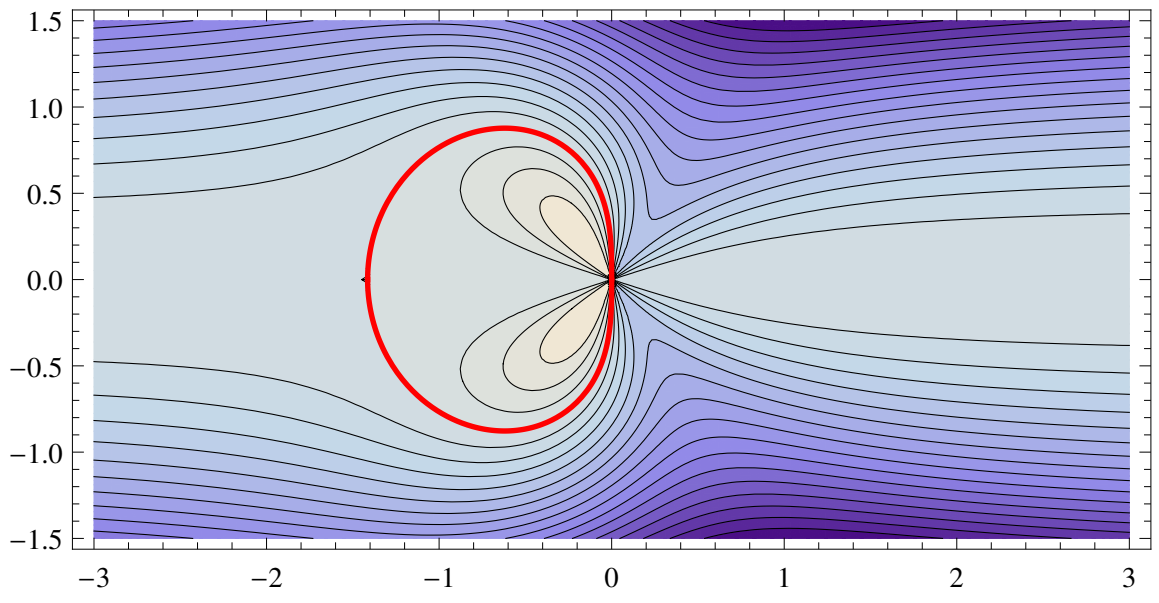


Figure 2: The streamlines in the comoving frame for the moving stresslet (Eq. (21)). The thick line shows the 'atmosphere' (closed streamline in the comoving frame).

2.1 Streamline pattern and atmosphere

We take the velocity field in a comoving frame to be

$$\mathbf{u}_{\text{comov}}(\mathbf{r}) = -\hat{\mathbf{z}} + \beta \left(1 - 3 \frac{zz}{\|\mathbf{r}\|^2} \right) \frac{\mathbf{r}}{\|\mathbf{r}\|^3}, \quad (21)$$

so that the stresslet is moving at unit speed in the $\hat{\mathbf{z}}$ direction, with a resulting apparent flow in the $-\hat{\mathbf{z}}$ direction. The streamfunction in the comoving frame is

$$\psi_{\text{comov}}(\rho, z) = -\frac{1}{2}\rho^2 - \beta \frac{z\rho^2}{\|\mathbf{r}\|^3}, \quad (22)$$

with

$$u_\rho = -\rho^{-1}\partial_z\psi, \quad u_z = \rho^{-1}\partial_\rho\psi. \quad (23)$$

Figure 2 shows the streamline pattern in the comoving frame (for $\beta = 1$), which suggests the presence of an atmosphere: a closed streamline in the comoving frame. We can find the equation for the atmosphere by solving $\psi_{\text{comov}} = 0$,

$$-\frac{1}{2} - \beta \frac{z}{(\rho^2 + z^2)^{3/2}} = 0, \quad (24)$$

so that

$$\rho_{\text{atm}}^2(z) = -z \left(z + (2\beta)^{2/3}(-1/z)^{1/3} \right). \quad (25)$$

Note that $\rho_{\text{atm}}(0) = \rho_{\text{atm}}(-\text{sign}(\beta)\sqrt{2|\beta|}) = 0$, which means the atmosphere extends from $z = 0$ to $z = -\text{sign}(\beta)\sqrt{2|\beta|}$. The atmosphere is plotted as a thick line in Fig. 2.

We also have an explicit expression for the volume of the atmosphere, for instance for $\beta > 0$:

$$V_{\text{atm}} = \int_{-\sqrt{2\beta}}^0 \pi \rho_{\text{atm}}^2(z) dz = \frac{8}{15} \sqrt{2\pi} |\beta|^{3/2}, \quad (26)$$

where the final expression is also valid for $\beta < 0$. The volume is useful for computing the transport due to particles trapped in the atmosphere.

2.2 Displacement for far field

Recall the definition of the two impact parameters, $a > 0$ and b (see Lin *et al.* (2011)). We set $U = \beta = 1$, and define

$$\mathbf{u}(\mathbf{r}, t) = \left(1 - 3 \frac{ZZ}{\|\mathbf{R}\|^2} \right) \frac{\mathbf{R}}{\|\mathbf{R}\|^3}, \quad (27)$$

where

$$\mathbf{R}(t) = (X, Y, Z(t)) = (x, y + a, z + b - t). \quad (28)$$

The stresslet starts at $(0, -a, -b)$ at $t = 0$ and proceeds to move in the positive z direction. The particle starts at $\mathbf{r} = 0$ and its motion takes place in the y - z plane. If the particle is far from the swimmer, then \mathbf{r} remains small throughout the trajectory, and we can expand to leading order in $\|\mathbf{r}\|$:

$$u_y = \frac{a(a^2 - 2(b-t)^2)}{H^5(a, b-t)} + O(\|\mathbf{r}\|), \quad u_z = \frac{(b-t)(a^2 - 2(b-t)^2)}{H^5(a, b-t)} + O(\|\mathbf{r}\|). \quad (29)$$

where the hypotenuse function is

$$H(a, b) := \sqrt{a^2 + b^2}. \quad (30)$$

At this order the particle feels a velocity field that is independent of its position. We can then solve for the particle motion by integrating $\dot{y} = u_y$ and $\dot{z} = u_z$:

$$y(t) = \frac{ab}{H^3(a, b)} - \frac{a(b-t)}{H^3(a, b-t)}, \quad (31a)$$

$$z(t) = \frac{H^2(a, \sqrt{2}b)}{H^3(a, b)} - \frac{H^2(a, \sqrt{2}(b-t))}{H^3(a, b-t)} \quad (31b)$$

valid to leading order in $\|\mathbf{r}\|$. Both coordinates achieve extrema at $t = b \pm \frac{1}{\sqrt{2}}a$, and $z(t)$ has an additional extremum at $t = b$. The fact that both coordinates achieve extrema at the same time is reflected by the two ‘cusps’ visible in Fig. 3. The coordinates of the two cusps are

$$y_{\text{cusp}} = \pm \frac{2}{3\sqrt{3}} a^{-1} + \frac{ab}{H^3(a, b)}, \quad z_{\text{cusp}} = -\frac{4}{3}\sqrt{\frac{2}{3}} a^{-1} + \frac{H^2(a, \sqrt{b})}{H^3(a, b)}. \quad (32)$$

After a time $t = \lambda$ (recall that $U = 1$, so $\lambda = U\tau = 1$), the net total displacement in each direction is $y(\lambda)$ and $z(\lambda)$. Examining Fig. 3 and using the location of the cusps (32) we find that the maximum displacement in y is bounded:

$$|y(\lambda)| \leq \frac{2}{3\sqrt{3}} a^{-1} + \frac{a|b|}{H^3(a, b)} \leq \frac{4}{3\sqrt{3}} a^{-1}. \quad (33)$$

The maximum displacement is achieved for $\lambda = \sqrt{2}a$, $b = \pm a/\sqrt{2}$. The displacement in z is also bounded:

$$|z(\lambda)| \leq -\frac{4}{3}\sqrt{\frac{2}{3}} a^{-1} + \frac{H^2(a, \sqrt{2}b)}{H^3(a, b)} \leq \left(\frac{4}{3}\sqrt{\frac{2}{3}} - 1\right) a^{-1}, \quad (34)$$

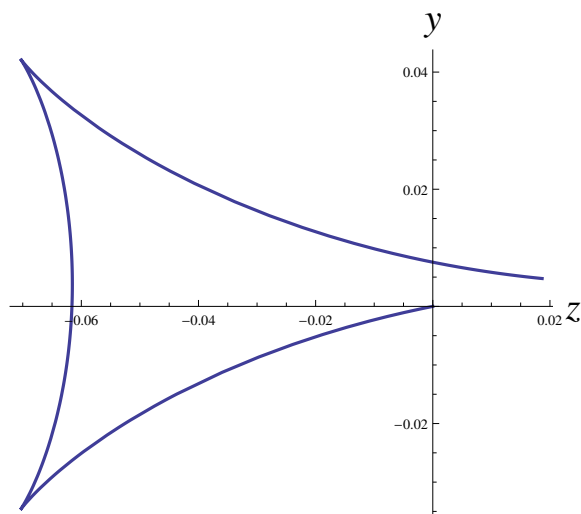


Figure 3: Particle trajectory for $a = 10$, $b = 50$, with t running from 0 to 150.

where the maximum is achieved for $\lambda = b = a/\sqrt{2}$.

In the limit of infinite path length, we have

$$|y(\lambda)| \sim \frac{ab}{H^3(a, b)}, \quad |z(\lambda)| \sim \frac{H^2(a, \sqrt{2}b)}{H^3(a, b)}, \quad \lambda \rightarrow \infty. \quad (35)$$

If we then take b to ∞ as well (the swimmer starts very far away), the displacement goes to zero. In this case, we have to expand the velocity field to next order to obtain the net displacement. It will not be necessary to do so here.

The total net displacement is

$$\Delta_\lambda(a, b) := \sqrt{y^2(\lambda) + z^2(\lambda)}. \quad (36)$$

To compute the effective diffusivity, we can evaluate the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^2 \Delta_\lambda^2(a, b) db d(\log a) = 4\lambda \quad (37)$$

whose integrand is plotted in Fig. 4. The resemblance to the numerical solution in Fig. 4(b) of Lin *et al.* (2011) is striking. The contributions to the integral (37) are $\frac{4}{3}\lambda$ from $y(\lambda)$ and $\frac{8}{3}\lambda$ from $z(\lambda)$. The displacement values for small a are not well predicted by this small displacement approach, but since the integral (4) downplays the importance of small a this will not lead to a large error.

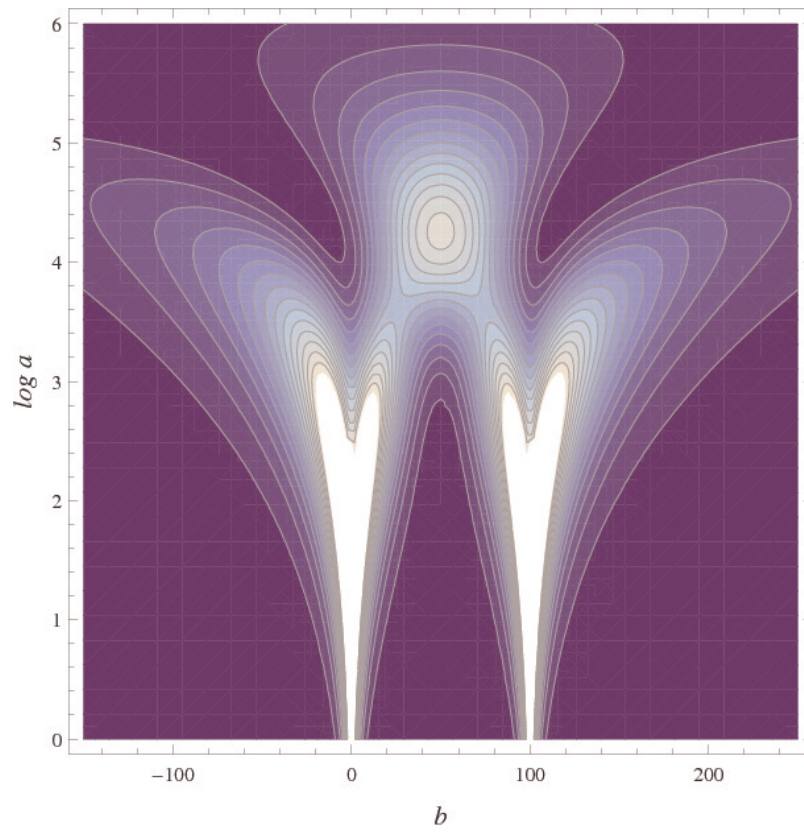


Figure 4: The integrand $a^2 \Delta_\lambda^2(a, b)$ for $\lambda = 100$.

Finally, we can use the result (37) in Eq. (2.6) of Lin *et al.* (2011) to compute the effective diffusivity:

$$D_{\text{eff}} = \frac{4}{3}\pi\beta^2 nU \quad (38)$$

where we restored all units (β has units of squared length times velocity). The path length λ drops out. In Lin *et al.* (2011) the definition of β is slightly different (replace our β by $\frac{3}{4}\beta\ell^2$ to recover their definition). Converting to their prefactor, we find a numerical coefficient of 2.356, whereas the numerical result in Lin *et al.* (2011) is 2.1 — a 10% difference, which is not bad for an analytic result! The difference is probably due to our large- a overestimating the displacement for small a .

References

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- Lin, Z., Thiffeault, J.-L., & Childress, S. (2011). *J. Fluid Mech.* **669**, 167–177.
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