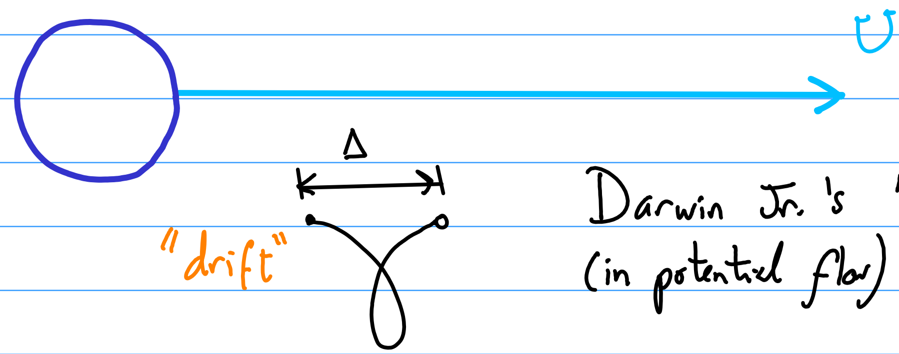
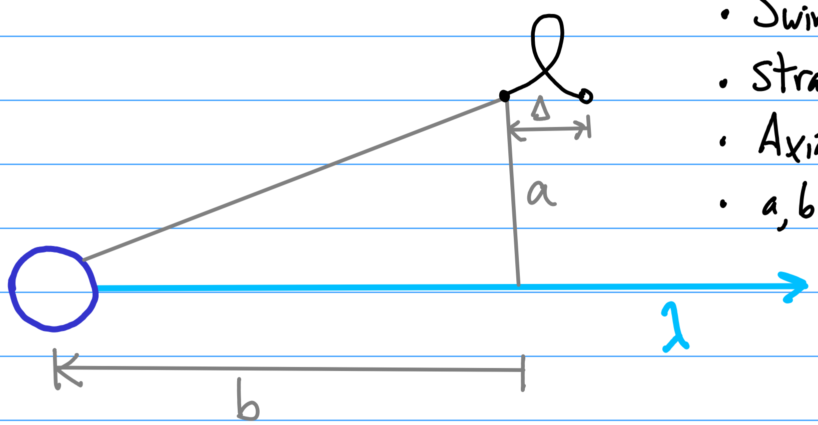
# Lecture 18: Displacements in inviscid flow

Single swimmer: take a cylinder




Darwin Jr.'s "elastica"
(in potential flow)

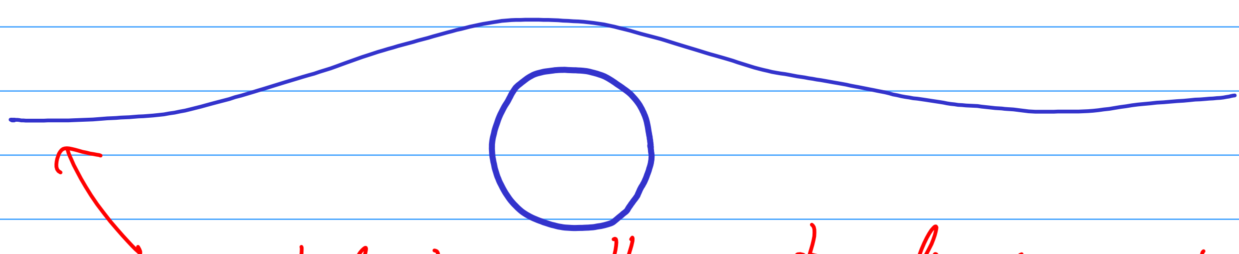
How do we compute $\Delta$?




- Swimming velocity is $U$ (const)
- Straight line for distance $\lambda$.
- Axially-symmetric, steady swimmer
- $a, b$ are "impact parameters" ($a > 0$)

Compute $\Delta_\lambda(a, b)$

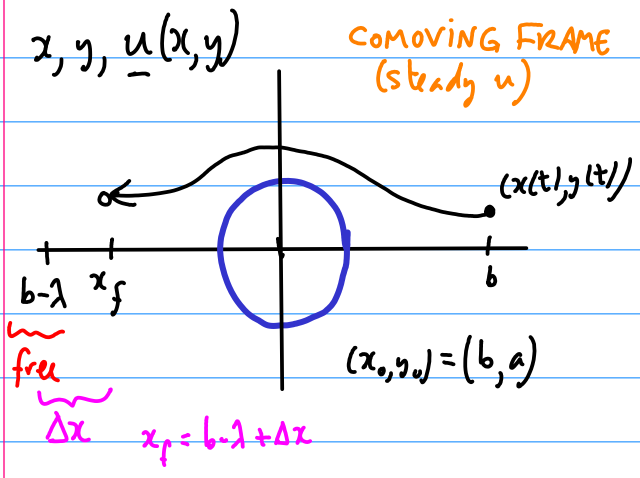
Far field: displacements tiny. Orbits almost closed.

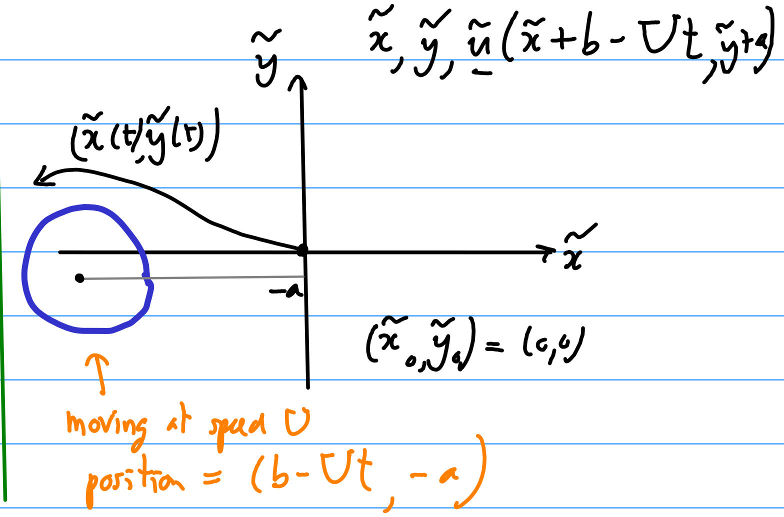


a particle stays on the same streamline in comoving frame

$\Delta y = 0$

Do 2D case (axisymmetric 3D similar):




$$\frac{d\tilde{x}}{dt} = \tilde{u}(\tilde{x}(t)+b-Ut, \tilde{y}(t)+a), \qquad x = \tilde{x}+b-Ut$$

$$\frac{dx}{dt} + U = \tilde{u}(x,y) \iff \boxed{\frac{dx}{dt} = -U + \tilde{u}(x,y) = u(x,y)}$$

autonomous (better!)

$$-\lambda + \Delta x = \int_0^{T=\lambda/U} u(x(t), y(t))\, dt$$

need both

For y: $\frac{dy}{dt} = \tilde{v}(x,y)$

Alternate form: $T = \frac{\lambda}{U} = \int_b^{x_f} \frac{dx}{u(x,y)}$, $x_f = b-\lambda+\Delta x$

how far particle moves when "free-streaming"

$$\frac{\lambda}{U} = \int_b^{b-\lambda+\Delta x} \frac{dx}{u} = -\int_b^{b-\lambda+\Delta x} \frac{dx}{|u|} = \int_{b-\lambda+\Delta x}^{b} dx/|u|$$

$$= \int_{b-\lambda}^{b} \frac{dx}{|u|} + \int_{b-\lambda+\Delta x}^{b-\lambda} \frac{dx}{|u|}$$

If particle doesn't move much and $|b-\lambda|$ "large", then $|u| \simeq U$

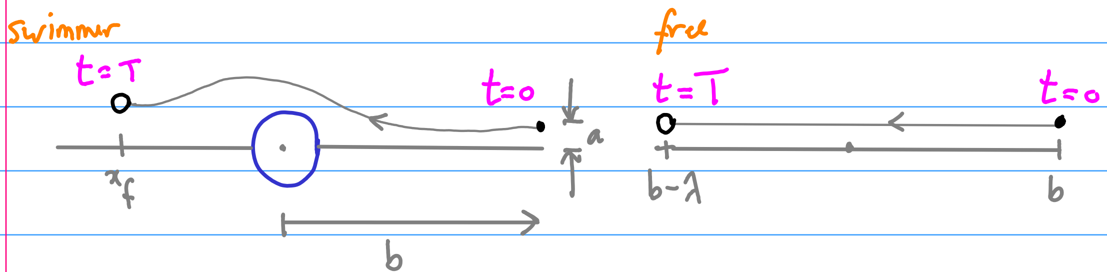
$$\frac{\lambda}{U} \simeq \int_{b-\lambda}^{b} \frac{dx}{|u|} - \frac{\Delta x}{U} \Leftrightarrow \Delta x = \int_{b-\lambda}^{b} \frac{dx}{|u|} - \frac{\lambda}{U}$$

$$\boxed{\Delta x \simeq \int_{b-\lambda}^{b} \left( \frac{1}{|u|} - \frac{1}{U} \right) dx}$$

Better form, since now can take $b \to \infty$, $b-\lambda \to -\infty$ if we want.

"Rayleigh form"

Intuitively, this formula measures the "lag" behind a free-streaming particle:




$$T = \frac{\lambda}{U}$$

2D incompressible: $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$$\psi(x_f, a + \Delta y) = \psi(b, a)$$ Same streamline

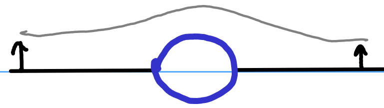
$$\psi(b - \lambda + \Delta x, a + \Delta y) = \psi(b, a)$$ solve for $\Delta y$, given $\Delta x$

If $|b-\lambda| \gg \Delta x$, $\boxed{\psi(b-\lambda, a+\Delta y) \simeq \psi(b,a)}$ solve for $\Delta y$

If also $\Delta y \ll a$, $\psi(b-\lambda, a) + \Delta y \underbrace{\partial_y \psi(b-\lambda, a)}_{u(b-\lambda, a)} \simeq \psi(b,a)$

$$\Delta y \simeq \frac{\psi(b,a) - \psi(b-\lambda, a)}{u(b-\lambda, a)}$$

Now for infinite $\lambda$, we have:



$$\Delta y = \frac{\psi(\infty, a) - \psi(-\infty, a)}{U} = 0!$$

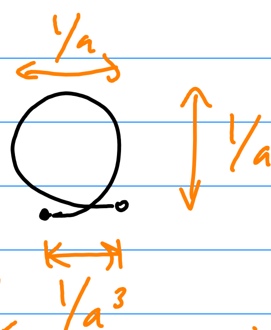
$$\boxed{\Delta y = 0}$$ for $\lambda \to \infty$, $b - \lambda \to -\infty$

Cylinder in potential flow:

$$\psi(x,y) = -Uy\left(1 - \frac{\ell^2}{x^2+y^2}\right)$$ Set $U = \ell = 1$

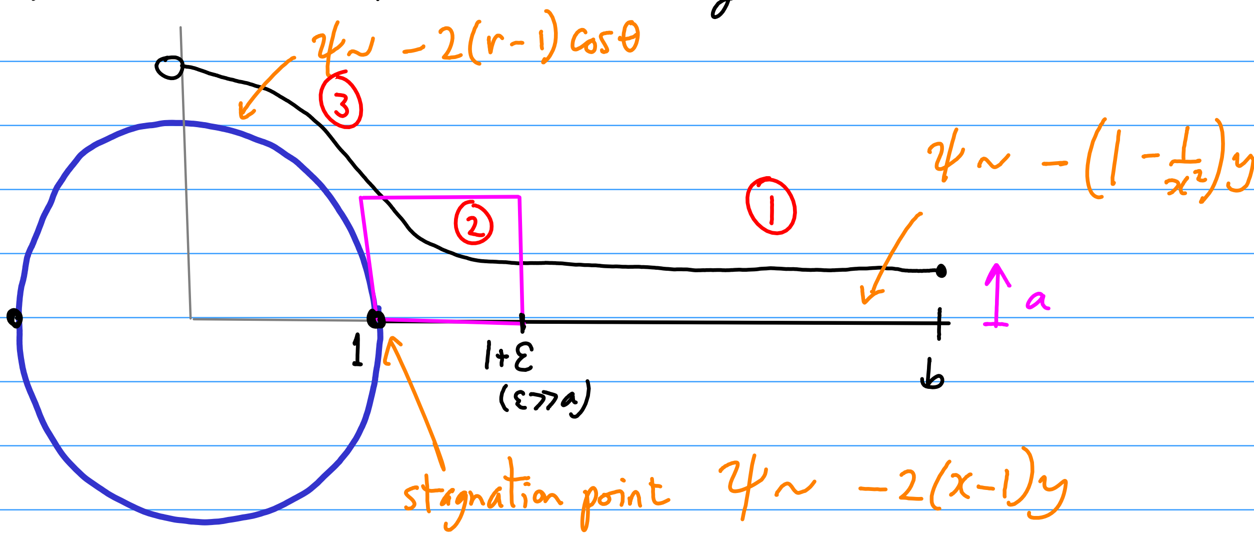
Far away, $\tilde{\psi} \sim \frac{y}{r^2}$, so $\tilde{u} \sim \frac{1}{r^2}$ in fixed frame

However, trajectories are almost closed,




Net result is $\Delta(a) \sim \frac{1}{a^3}$ Much smaller than overall excursion!

The limit $a \ll 1$ is more interesting:



free flow ↻ (to the left)

Need to calculate $\int \left(\frac{1}{u}+1\right)dx$ over each region ①, ②, ③.

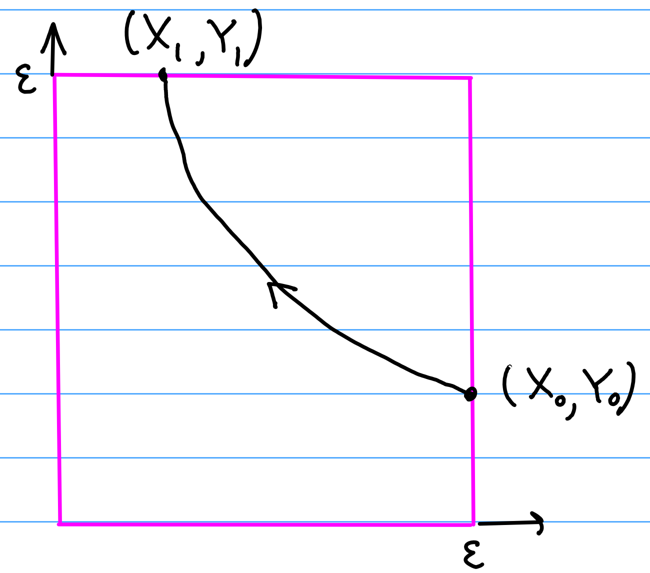
**Region 1:** $\psi_0 = \psi(b,a) = -(1-b^{-2})a$

$$u = -(1-x^{-2}), \quad T_1 = \int_b^{1+\varepsilon} \left(\frac{1}{u}+1\right)dx = \int_b^{1+\varepsilon} \frac{dx}{1-x^2}$$

transit time

After using $\varepsilon \ll 1$, $b \gg 1$: $T_1 \simeq \frac{1}{2}\log(2/\varepsilon) + \varepsilon/4 - b^{-1} + O(\varepsilon^2, b^{-2})$

**Region 2:**




$X = x-1, \quad Y = y$

$\psi = -2XY$

neglect

At $X_0, Y_0$,

$\psi = -2X_0Y_0 = -(1-b^{-2})a$

ε

$\Rightarrow Y_0 = \frac{a}{2\varepsilon}$

$(X_0, Y_0) = (\varepsilon, a/2\varepsilon)$

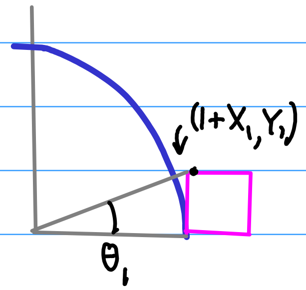
But also $Y_1 = \varepsilon$, so $X_1 = a/2\varepsilon$.

small

$$T_2 = \int_{X_0}^{X_1} \left(\frac{1}{u}+1\right)dx = \int_{\varepsilon}^{a/2\varepsilon} \left(\frac{1}{(-2X)}+1\right)dx = -\frac{1}{2}\log\left(\frac{a}{2\varepsilon^2}\right) + \frac{a}{2\varepsilon} - \varepsilon$$

$$u = -1 + \frac{\cos 2\theta}{r^2}$$

**Region 3:**




$$T_3 = \int_{\theta_1}^{\pi/2} \left(\frac{1}{u} + 1\right) \underbrace{\frac{dx}{d\theta}}_{-r\sin\theta} d\theta$$

$$= \frac{1}{2} \int_{\theta_1}^{\pi/2} \frac{\cos 2\theta}{\sin\theta}\, d\theta \quad , \ \theta_1 \text{ small.}$$

$$T_3 \simeq -1 + \frac{1}{2}\log 2 - \frac{1}{2}\log\theta_1 + O(\theta_1^2)$$

$$\tan\theta_1 = \frac{Y_1}{1+X_1} = \frac{\varepsilon}{1+a/\varepsilon} \simeq \varepsilon(1 - a/\varepsilon) = \varepsilon - a$$

$$\therefore\ T_3 \simeq -1 + \frac{1}{2}\log 2 - \frac{1}{2}\log\varepsilon + \frac{1}{2}\frac{a}{\varepsilon} + O((a/\varepsilon)^2)$$

Add everything together:

divergent $\log\varepsilon$ terms cancel

$$T = T_1 + T_2 + T_3 = \left(\frac{1}{2}\log(2/\cancel{\varepsilon}) - 6^{-1}\right) + \left(-\frac{1}{2}\log\left(\frac{a}{\cancel{2\varepsilon}}\right)\right)$$

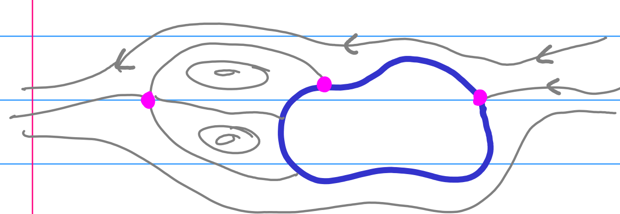
$$+ \left(-1 + \frac{1}{2}\log 2 - \cancel{\frac{1}{2}\log\varepsilon}\right)$$

$$T = \underbrace{-\frac{1}{2}\log a}_{} - 1 + \frac{3}{2}\log 2 - 6^{-1}$$ to leading order.

Dominant term for small a.
Comes only from region 2, near stagnation point.

$T \to \infty$ for $a \to 0$. particle gets stuck!

The total drift is given by $2T$, since the body is fore-aft symmetric.

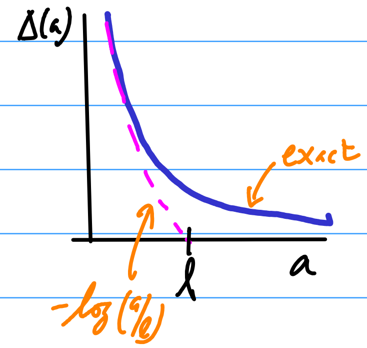


In general, the coefficient of $\log a$ is given by summing over the linearization coeffs for each (hyperbolic) stagnation pt encountered. (not true for no-slip!)

Note that to pick up the $-\log a$ contribution, the target particle must come in the vicinity of the stagnation points

$$\Delta_\lambda(a,b) = \begin{cases} -\log a & , \quad 0 \leq b \leq \lambda \\ \text{(neglect)} & , \quad \text{otherwise} \end{cases}$$

Cylinder. $\kappa \simeq \dfrac{2Un}{\cancel{\lambda}}\cancel{\lambda}\displaystyle\int_0^{\ell} \log^2\left(\frac{a}{\lambda}\right) da$



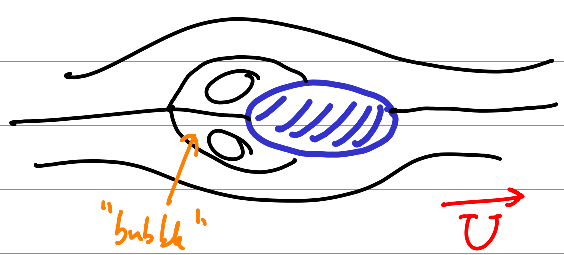

$$\int \log^2 x \, dx = x\log^2 x - 2x\log x + 2x$$

$$\int_0^1 \log^2 x \, dx = 2$$ (numerical answer: 2.37)

$$\boxed{\kappa \simeq Un\ell^3}$$ (numerical: $\kappa = 1.19\, Un\ell^3$)

Note that this is completely independent of $\lambda$!

Another example: consider a swimmer with a bubble "wake":



If a particle is trapped in the bubble, moves by $\lambda$.

$$\Delta_\lambda(a,b) = \begin{cases} \lambda, & \text{particle inside bubble} \\ \text{neglect}, & \text{otherwise} \end{cases}$$

Total volume of bubble

$$6\kappa = \frac{2Un}{\cancel{\lambda}} \int_{\text{inside bubble}} \lambda^{\cancel{2}} \, da\, db = U n \lambda V_{\text{bubble}}$$

The 2 goes away since $2\, da\, db$ is volume element

$$\boxed{\kappa = \frac{1}{6} U n \lambda V_{\text{bubble}}}$$

$$V_{\text{bubble}} = \text{area in 2D} = \text{volume in 3D}$$

Now this depends on path length $\lambda$. This can be much larger than for untrapped fluid. Real swimmer probably in between