# Lecture 17: Biomixing, part 2: Effective diffusivity

In the previous lecture we derived an expression for the ditribution of number of interactions $m$ with a sphere of radius $R$:

$$\mathbb{P}\{M_t = m\} \simeq \frac{1}{\sqrt{2\pi \operatorname{Var} M_t}} \, \mathrm{e}^{-(m-\langle M_t \rangle)^2/2 \operatorname{Var} M_t}, \qquad \langle M_t \rangle \gg 1, \tag{1}$$

where the expected number of interactions is

$$\langle M_t \rangle = n \left\{ V_{\mathrm{swept}}(R,\lambda)\,(t/\tau) + V_{\mathrm{sph}}(R) \right\}, \tag{2}$$

with $n$ the number density of swimmers, $t$ the time elapsed, $\tau$ the duration of a path, $\lambda$ the length of a path, and $V_{\mathrm{swept}}(R,\lambda)$ and $V_{\mathrm{sph}}(R)$ the volume of a cylinder and sphere.

Now that we've examined how often swimmers interact with a sphere of radius $R$ centered around a target particle, we will look at how the particle gets displaced. Figure 1 shows the setup of an interaction. Since the system is homogeneous and isotropic, only two 'impact parameters' $a$ and $b$ are needed to describe an interaction. These are depicted in the figure: here $C$ is the point along the line of motion that is closest to the initial position of the particle, and $a \in [0, R]$ is this closest distance. The parameter $b \in [-R, \lambda + R]$ is the distance from $C$ to the initial position of the swimmer. A negative value of $b$ means the swimmer started its path beyond the point $C$.

Following Lin *et al.* (2011), we start from a distribution of displacements $\Delta_\lambda(a, b)$ induced by a single swimmer. Here the impact parameters $a$ and $b$ describe the encounter between the swimmer and a target particle, and $\lambda$ is the path length of swimming (Fig. 1). Each time a swimmer enters the interaction sphere we have an 'encounter,' which causes a displacement of the target particle; thus, after $m$ encounters, the $x$ displacement is

$$X_m = \sum_{k=1}^{m} \Delta_\lambda(a_k, b_k) \cos\psi_k \tag{3}$$

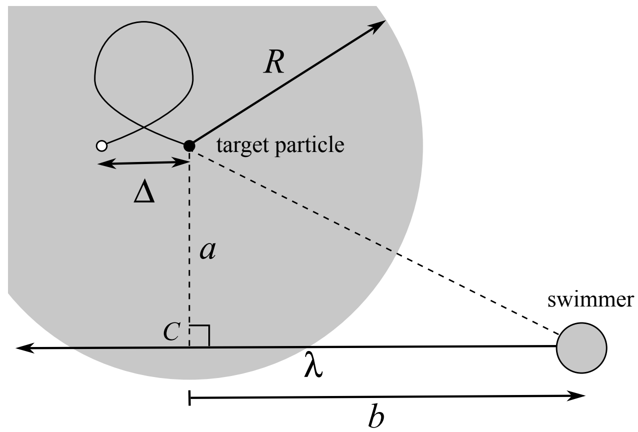


Figure 1: Definition of impact parameters $a$ and $b$, displacement $\Delta = \Delta_\lambda(a,b)$, and swimming path length $\lambda$. In this picture the parameter $b$ is positive; negative $b$ corresponds to the swimmer starting its trajectory past the point $C$ of smallest initial perpendicular distance to the line of motion. The filled dot is the initial position of the target particle and the hollow dot is its final position after the swimmer has moved by a distance $\lambda$. The 'interaction sphere' of radius $R$ is also shown. (After Lin *et al.* (2011).)

where each encounter has random i.i.d. values of the impact parameters $a_k$ and $b_k$ and angle $\psi_k$. We select the $X$ displacement here, but by isotropy the statistics in any direction are the same.

The probability density of $X_m$ can be related to that of $X_t$, the $x$ displacement after a time $t$, by first observing that $\mathbb{P}\{X_m \in [x, x + \mathrm{d}x]\} = \mathbb{P}\{X_t \in [x, x + \mathrm{d}x] \mid M_t = m\}$, and

$$\begin{aligned}\mathbb{P}\{X_t \in [x, x + \mathrm{d}x]\} &= \sum_{m=0}^{\infty} \mathbb{P}\{X_t \in [x, x + \mathrm{d}x]\,,\, M_t = m\} \\ &= \sum_{m=0}^{\infty} \mathbb{P}\{X_t \in [x, x + \mathrm{d}x] \mid M_t = m\}\, \mathbb{P}\{M_t = m\}, \qquad (4)\end{aligned}$$

where $\mathbb{P}\{M_t = m\}$ is the probability of getting $m$ encounters in time $t$. If the latter is sharply peaked, such as in the Gaussian limit (1), then we can just use $m \simeq \langle M_t \rangle$. But for now let us focus on $\mathbb{P}\{X_m \in [x, x + \mathrm{d}x]\}$.

We wish to derive the PDF of the total $x$ displacement $X_m$, assuming that the random variables $a_k$, $b_k$, $\psi_k$ are independent for different $k$ and identically distributed, with probabiliy densities $\rho_{ab}(a_k, b_k)$ and $\rho_\psi(\psi_k)$. Because of isotropy, the angular

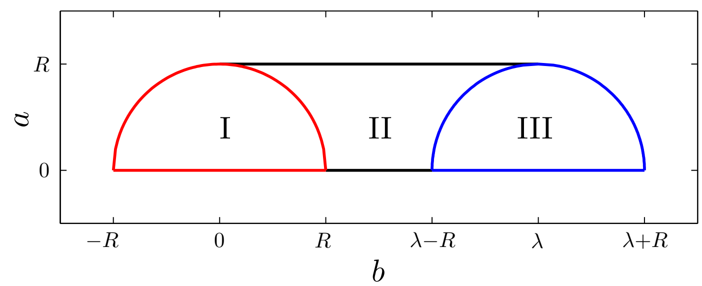


Figure 2: The domain $\Omega_{ab} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ of the impact parameters $a$ and $b$ for fixed path length $\lambda$ (see Fig. 1). Region I corresponds to swimmers that start their path inside the interaction sphere; swimmers in Region II cross the sphere completely; swimmers in Region III finish their path inside the sphere. Note that the figure depicts $\lambda > 2R$, but all the formulas hold for $\lambda < 2R$ as well, when regions I and III overlap because some trajectories both start and finish inside the sphere.

variables have simple densities:

$$\rho_\psi(\psi) = 1/2\pi, \quad \Omega_\psi = [0, 2\pi] \quad \text{(2D)}; \qquad \rho_\psi(\psi) = \tfrac{1}{2}\sin\psi, \quad \Omega_\psi = [0, \pi] \quad \text{(3D)}, \tag{5}$$

for $\psi \in \Omega_\psi$. In two dimensions, the joint density $\rho_{ab}(a, b)$ is uniform over the domain $\Omega_{ab} = \{0 \le a \le R,\ -\sqrt{R^2 - a^2} \le b \le \lambda + \sqrt{R^2 - a^2}\}$ depicted in Fig. 2. These are the values of $a$ and $b$ for which a swimmer's straight path intersects the interaction sphere. After normalizing, we find the density

$$\rho_{ab}(a, b) = 2/V_{\text{swept}}(R, \lambda) \quad \text{(2D)}. \tag{6}$$

In three dimensions, the domain in Fig. 2 is interpreted as a surface of revolution about $a = 0$, leading to the density

$$\rho_{ab}(a, b) = 2\pi a/V_{\text{swept}}(R, \lambda) \quad \text{(3D)}. \tag{7}$$

For both the 2D and 3D cases, $\rho_{ab}(a, b)$ is then normalized such that

$$\int_{\Omega_{ab}} \rho_{ab}(a, b)\,\mathrm{d}a\,\mathrm{d}b = \int_0^R \int_{-\sqrt{R^2-a^2}}^{\lambda+\sqrt{R^2-a^2}} \rho_{ab}(a, b)\,\mathrm{d}b\,\mathrm{d}a = 1. \tag{8}$$

We have the convenient forms

$$\langle M_t \rangle \rho_{ab} \simeq 2nt/\tau \quad \text{(2D)}; \qquad \langle M_t \rangle \rho_{ab} \simeq 2\pi a n t/\tau \quad \text{(3D)}, \tag{9}$$

in terms of the expected values (2). These are valid for $t \gg \tau$, so we can neglect the extra added spherical volume in (2).

We can now compute the effective diffusivity. We have of course $\langle X_M \rangle = 0$ because of isotropy. The variance is then

$$\langle X_m^2 \rangle = \sum_{k=1}^{m} \langle \Delta_\lambda^2(a_k, b_k) \cos^2 \psi_k \rangle = m \langle \Delta_\lambda^2(a, b) \rangle \langle \cos^2 \psi \rangle \tag{10}$$

since the variables are i.i.d. The angular average is

$$\langle \cos^2 \psi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \psi \, \mathrm{d}\psi = \tfrac{1}{2} \qquad \text{(2D)}; \tag{11}$$

$$\langle \cos^2 \psi \rangle = \frac{1}{2} \int_0^{\pi} \cos^2 \psi \sin \psi \, \mathrm{d}\psi = \tfrac{1}{3} \qquad \text{(3D)}. \tag{12}$$

So now we define the *effective diffusivity* $D$

$$\langle X_m^2 \rangle = \frac{m}{d} \langle \Delta_\lambda^2(a, b) \rangle = 2Dt \tag{13}$$

where $d$ is the dimension of space. We have finally

$$D = \frac{m}{2dt} \langle \Delta_\lambda^2(a, b) \rangle, \tag{14}$$

where

$$\langle \Delta_\lambda^2(a, b) \rangle = \int_{\Omega_{ab}} \rho_{ab}(a, b) \, \Delta_\lambda^2(a, b) \, \mathrm{d}a \, \mathrm{d}b. \tag{15}$$

Assume now that $m = \langle M_t \rangle$, which wil be satisfied if there are many encounters. Then using (9) we find

$$D = \frac{n}{2\tau} \int_{\Omega_{ab}} \Delta_\lambda^2(a, b) \, \mathrm{d}a \, \mathrm{d}b, \qquad \text{(2D)}; \tag{16}$$

and

$$D = \frac{\pi n}{3\tau} \int_{\Omega_{ab}} \Delta_\lambda^2(a, b) \, a \, \mathrm{d}a \, \mathrm{d}b, \qquad \text{(3D)}. \tag{17}$$

where recall that $\tau = \lambda / U$ is the path length of swimming. Notice the extra $a$ in the 3D integrand, due to the fact that there is a 'ring' of points a distance $a$ from the target. This extra $a$ will modify the dependence in 2D and 3D quite dramatically.

So far everything is quite general, as long as the density of swimmers is low enough. In the next lecture we will discuss the most crucial part: how to model $\Delta_\lambda(a, b)$. This depends heavily on the kind of swimmer and the type of fluid.

# References

Lin, Z., Thiffeault, J.-L., & Childress, S. (2011). *J. Fluid Mech.* **669**, 167–177. http://arxiv.org/abs/1007.1740.

Thiffeault, J.-L. & Childress, S. (2010). *Phys. Lett. A,* **374**, 3487–3490.