Lecture 16: Biomixing, part 1: hitting distribution

We use a simple model described by Thiffeault & Childress (2010) and refined by Lin *et al.* (2011), which is convenient for visualization and for taking limits. We assume there are N swimmers in a volume V, so the number density of swimmers is $n = N/V$. Initially, each swimmer travels at a speed U in a uniform random direction. They keep moving along a straight path for a time τ, so that each traces out a segment of length $\lambda = U\tau$. After this a new direction is chosen randomly and uniformly, and the process repeats — each swimmer again moves along a straight path of length λ. Though far from realistic, this model captures many essential features of the system, as found by Thiffeault & Childress (2010); Lin *et al.* (2011) and as we'll explore further in this paper. We will discuss later how this model could be refined.

We wish to follow the displacement of an arbitrary 'target fluid particle.' The swimmers are all simultaneously affecting this fluid particle, but in practice only the closest swimmers significantly displace it. It is thus convenient to introduce an imaginary 'interaction sphere' of radius R centered on the target fluid particle, and count the number M_t of 'interactions,' that is the number of times a swimmer enters this sphere. (Our treatment applies to two-dimensional systems simply by changing 'sphere' to 'disk' and 'volume' to 'area.') Figure 1 illustrates the situation.

Each time a swimmer enters the interaction sphere, the target particle is displaced by some distance. We will address this in the next section and see how to sum the displacements due to many swimmers to obtain the distribution of the net displacement x. For now, let us find the distribution of M_t, the number of times a swimmer crosses the interaction sphere during a time t.

The probability that the swimmer starts inside a small volume $\mathrm{d}V$ is $\mathrm{d}V/V$, where V is the total volume. The probability of a swimmer actually starting inside the interaction sphere is then $V_{\mathrm{sph}}(R)/V$, where $V_{\mathrm{sph}}(R)$ is the volume of a sphere of radius R. (We assume the interaction sphere fits completely within the volume V.) We define the event

$$H_t = \text{a swimmer crosses the interaction sphere once during time } t < \tau \ (= \lambda/U), \tag{1}$$

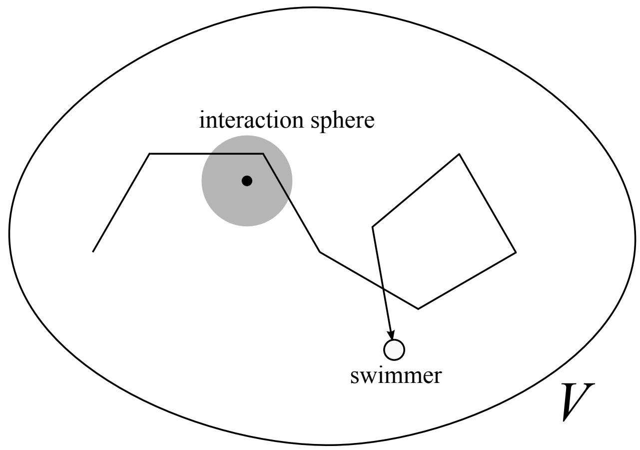

Figure 1: A swimmer moving inside a volume V along a series of straight paths, each of length λ and in a uniform random direction. The interaction sphere around the target particle (black dot) is shown in gray. Here the swimmer 'interacts' twice with the target particle, since two of its paths intersect the sphere.

that is, the center of the swimmer is inside the interaction sphere at some point while traveling on a straight path of length $Ut < \lambda$, where U is the uniform speed of a swimmer. To determine the probability of H_t, observe that because of the homogeneity and isotropy of the swimmers this probability is proportional to the volume swept out by the interaction sphere if it moves a distance Ut, with $0 \le t < \tau$:

$$p_t := \mathbb{P}(H_t) = V_{\text{swept}}(R, \lambda)/V, \qquad V_{\text{swept}}(R, \lambda) := V_{\text{cyl}}(R, \lambda) + V_{\text{sph}}(R), \tag{2}$$

where

$$V_{\text{cyl}}(R, \lambda) := \begin{cases} 2R\lambda, \\ \pi R^2 \lambda, \end{cases} \qquad V_{\text{sph}}(R) := \begin{cases} \pi R^2, & \text{(2D);} \\ \frac{4}{3}\pi R^3, & \text{(3D);} \end{cases} \tag{3}$$

are respectively the volume of the cylinder of radius R swept out in time t and the volume of the interaction sphere, which gives the probability that a swimmer starts inside the interaction sphere. This assumes that all points on the interaction sphere's surface are at least a distance λ from the boundary of V.

For N swimmers, let M_t be the total number of interactions with the sphere during time t. In Appendix we use a generating function approach to find the probabiliy distribution of M_t, and show that

$$\langle M_t \rangle = n \left\{ V_{\text{swept}}(R, \lambda) \, (t/\tau) + V_{\text{sph}}(R) \right\} \tag{4}$$

where $n = N/V$ is the number density of swimmers. In this form we can take the limits $N \to \infty$ and $V \to \infty$ while keeping n constant, which doesn't change the expectation value.

Also from Appendix , the variance of M_t is

$$\operatorname{Var} M_t = N \left(p_\tau (1 - p_\tau) \, (t/\tau) + \tfrac{1}{3} p_\tau^2 - \tfrac{1}{3} (V_{\mathrm{sph}}(R)/V)(2p_\tau + 2(V_{\mathrm{sph}}(R)/V) - 3) \right) \quad (5)$$

where $V_{\mathrm{sph}}(R)$ is the volume (or area) of the interaction sphere. Any term in (5) quadratic in $V_{\mathrm{sph}}(R)$ or p_τ will vanish as $V \to \infty$, and we are left with

$$\operatorname{Var} M_t \sim \langle M_t \rangle, \qquad V \to \infty. \quad (6)$$

For large $\langle M_t \rangle$ we thus expect that a typical value of M_t will be very close to the mean, since $\langle M_t \rangle / \sqrt{\operatorname{Var} M_t}$ is small. In that case, the central limit theorem applies (M_t is the sum of i.i.d. random variables) and we have the Gaussian approximation

$$\mathbb{P}\{M_t = m\} \simeq \frac{1}{\sqrt{2\pi \operatorname{Var} M_t}} \, \mathrm{e}^{-(m - \langle M_t \rangle)^2 / 2 \operatorname{Var} M_t}, \qquad \langle M_t \rangle \gg 1, \quad (7)$$

with $\langle M_t \rangle$ defined in (4). The mean and variance equations (4) and (5) are exact as long as the interaction sphere is more than a path length λ away from the boundary of V; equation (7) further requires $\langle M_t \rangle \gg 1$, which typically happens for long times. Figures 2(a)–2(b) show the convergence to a Gaussian distribution for numerical simulations of moving swimmers, in 2D and 3D.

Appendix: Generating function approach for random phases

The generating function of a sequence $\{a_n\}$ is defined as Feller (1968)

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n. \quad (8)$$

Now let a_n give the probability of having n events H_t. For a single swimmer moving for a time $t < \tau$, we can only have $n = 0$ or 1 events, with probability $a_0 = (1 - p_t)$ and $a_1 = p_t$; hence,

$$G_t(x) = a_0 + a_1 x = (1 - p_t) + p_t \, x \,, \qquad t < \tau. \quad (9)$$

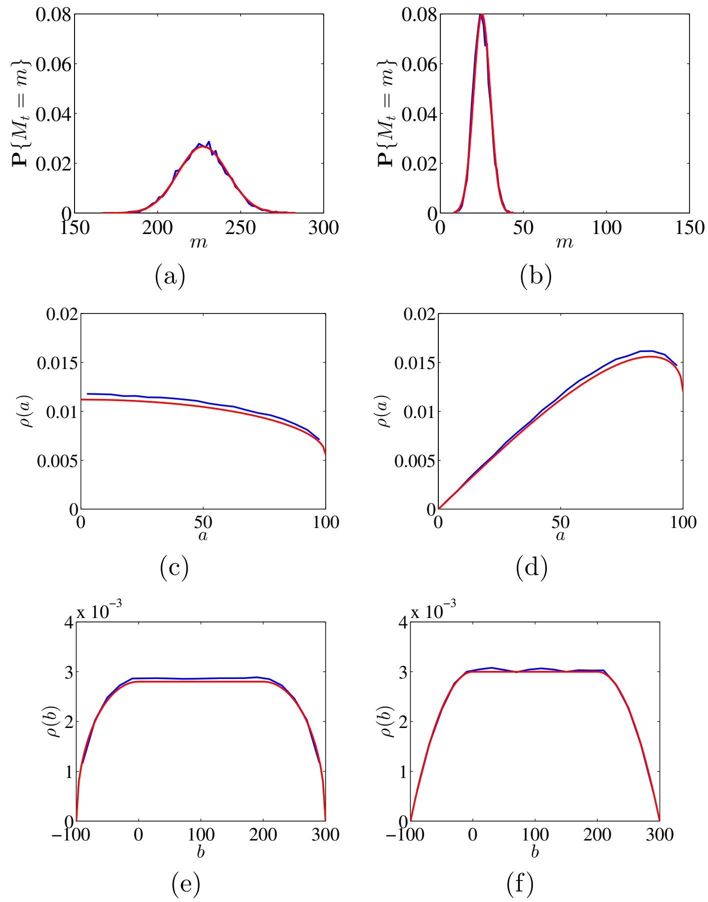

Figure 2: Probability distribution function for $N = 1000$ swimmers to enter the interaction sphere $M_t = m$ times, in (a) 2D and (b) 3D. The interaction sphere has radius $R = 100$, the path length $\lambda = 200$, the total volume is a sphere of radius $L = 1000$, and the number of steps is $k = \lfloor Ut/\lambda \rfloor = 10$. Shown in red is the Gaussian approximation (7). (c)–(d) Marginal probability densities $\rho(a)$ in 2D and 3D, respectively. (e)–(f) Marginal probability densities $\rho(b)$ in 2D and 3D, respectively.

The expected number of events is $\langle M_t \rangle = G_t'(1) = p_t$. If the swimmer moves for a time $t = k\tau$, $k \in \mathbb{Z}^+$, the total number of events is the sum of the events at each interval τ. The resulting generating function is then $G_\tau^k(x)$, assuming that the swimmer starts on its first path at $t = 0$. More generally, if the swimmer has already started on a path before $t = 0$, then

$$G_t(x) = G_{\tau_0}(x)\, G_\tau^{k_{t,\tau_0}}(x)\, G_{\tau_1}(x) \tag{10}$$

where $\tau_0 + \tau k_{t,\tau_0} + \tau_1 = t$, $k_{t,\tau_0} = \lfloor (t - \tau_0)/\tau \rfloor$, and $0 \le \tau_i < \tau$. The two τ_i pieces account for the partial paths traversed at the beginning and at the end of the motion. We take $\tau_0 \in [0, \tau)$ to be a uniformly-distributed random variable; τ_1 then follows from $\tau_1 = t - \tau_0 - \tau k_{t,\tau_0}$.

Now write $p_t = \alpha t + \beta$, where the constants α and β come from (2). The expected number of events H_t is

$$\begin{aligned}\langle M_t \rangle = \langle p_{\tau_0} + k_{t,\tau_0} p_\tau + p_{\tau_1} \rangle &= \langle \alpha(\tau_0 + \tau k_{t,\tau_0} + \tau_1) + (k_{t,\tau_0} + 2)\beta \rangle \\ &= \alpha t + \beta(2 + \langle k_{t,\tau_0} \rangle).\end{aligned}$$

To compute $\langle k_{t,\tau_0} \rangle$, let $t/\tau = \ell + \delta$, $\ell = \lfloor t/\tau \rfloor$, $\delta \in [0, 1)$. Then $\langle k_{t,\tau_0} \rangle = \langle \lfloor (t - \tau_0)/\tau \rfloor \rangle = \ell + \langle \lfloor \delta - \tau_0/\tau \rfloor \rangle$, with $|\delta - \tau_0/\tau| < 1$, and

$$\langle \lfloor (\delta - \tau_0)/\tau \rfloor \rangle = \frac{1}{\tau} \int_0^\tau \lfloor \delta - \tau_0/\tau \rfloor \, \mathrm{d}\tau_0 = \int_0^1 \lfloor \delta - \xi \rfloor \, \mathrm{d}\xi = \int_\delta^1 (-1)\, \mathrm{d}\xi = \delta - 1.$$

Thus,

$$\langle k_{t,\tau_0} \rangle = \ell + \delta - 1 = t/\tau - 1, \tag{11}$$

and we finally conclude

$$\langle M_t \rangle = (\alpha\tau + \beta)\, t/\tau + \beta = p_\tau\, (t/\tau) + \beta. \tag{12}$$

The extra β at the end arises from the possibility of swimmers starting inside the interaction sphere at $t = 0$.

We can also compute the variance exactly. For a single swimmer,

$$\operatorname{Var} M_t = G_t''(1) + G_t'(1) - [G_t'(1)]^2 = p_t - (p_t)^2 = p_t(1 - p_t), \qquad t < \tau, \tag{13}$$

and for longer time

$$\begin{aligned}\operatorname{Var} M_t &= \langle p_{\tau_0}(1 - p_{\tau_0}) + k_{t,\tau_0}\, p_\tau(1 - p_\tau) + p_{\tau_1}(1 - p_{\tau_1}) \rangle \\ &= \langle M_t \rangle - \langle p_{\tau_0}^2 + k_{t,\tau_0}\, p_\tau^2 + p_{\tau_1}^2 \rangle \le \langle M_t \rangle.\end{aligned}$$

Now we need to compute the expectation value of this over τ_0. This is a slightly tedious calculation which we do not present; the final result is

$$\operatorname{Var} M_t = p_\tau(1 - p_\tau)\,(t/\tau) + \tfrac{1}{3}\alpha^2\tau^2 + \beta(1-\beta)\,. \tag{14}$$

For N swimmers, because we are still summing the number of displacements the generating function will be the product of several copies of (10):

$$G_t^N(x) = \prod_{j=1}^{N} G_{\tau_{0,j}}(x)\, G_\tau^{k_{t,\tau_{0,j}}}(x)\, G_{t-\tau_{0,j}-\tau k_{t,\tau_{0,j}}}(x) \tag{15}$$

where each swimmer has its own random initial partial path $\tau_{0,j}$. The probability distribution will thus be a convolution of all these generating functions, and the expected value and variance will add up. The net result is to multiply the expected number of events (12) and its variance (14) by N. After substituting the value of α and β from $p_t = \alpha t + \beta$ and (2) and using $n = N/V$, we obtain equations (4) and (5).

References

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, volume 1. New York: John Wiley & Sons, third edition.

Lin, Z., Thiffeault, J.-L., & Childress, S. (2011). *J. Fluid Mech.* **669**, 167–177. http://arxiv.org/abs/1007.1740.

Thiffeault, J.-L. & Childress, S. (2010). *Phys. Lett. A*, **374**, 3487–3490.