Lecture 15: Homogenization Theory

I. MULTISCALE EXPANSION AND HOMOGENIZATION

We start with the advection–diffusion equation,

$$\partial_t \varphi(t, \boldsymbol{r}) + \boldsymbol{u}(\boldsymbol{r}) \cdot \nabla_{\boldsymbol{r}} \varphi(t, \boldsymbol{r}) = D \Delta_{\boldsymbol{r}} \varphi(t, \boldsymbol{r}). \tag{I.1}$$

Assume typical lengthscale of $\boldsymbol{u}$ is ℓ, and that the initial condition varies on a scale L that is large with respect to ℓ. Define $\varepsilon = \ell/L \ll 1$. We write $\varphi(0, \boldsymbol{r}) = \varphi_0(\varepsilon\, \boldsymbol{r})$.

Now introduce the large scale and slow time,

$$\boldsymbol{R} = \varepsilon\, \boldsymbol{r}, \qquad T = \varepsilon^2 t\,, \tag{I.2}$$

and assume that the concentration depends on these scales,

$$\varphi(t, \boldsymbol{r}) = \varphi^\varepsilon(T, \boldsymbol{r}, \boldsymbol{R}). \tag{I.3}$$

Using $\partial_t \to \varepsilon^2\, \partial_T$, $\nabla_{\boldsymbol{r}} \to \nabla_{\boldsymbol{r}} + \varepsilon\, \nabla_{\boldsymbol{R}}$, Eq. (I.1) becomes

$$\mathcal{L}\varphi^\varepsilon + \varepsilon^2\, \partial_T \varphi^\varepsilon + \varepsilon\, \boldsymbol{u}(\boldsymbol{r}) \cdot \nabla_{\boldsymbol{R}} \varphi^\varepsilon = 2\varepsilon\, D \nabla_{\boldsymbol{r}} \cdot \nabla_{\boldsymbol{R}} \varphi^\varepsilon + \varepsilon^2 D \Delta_{\boldsymbol{R}} \varphi^\varepsilon \tag{I.4}$$

where the velocity field is assumed to only depend on the short lengthscale $\boldsymbol{r}$, and we have defined the linear operator

$$\mathcal{L} := -D\Delta_{\boldsymbol{r}} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{r}}\,. \tag{I.5}$$

We expand the concentration in a power series in ε,

$$\varphi^\varepsilon(T, \boldsymbol{r}, \boldsymbol{R}) = \varphi^{(0)}(T, \boldsymbol{r}, \boldsymbol{R}) + \varepsilon\, \varphi^{(1)}(T, \boldsymbol{r}, \boldsymbol{R}) + \ldots \tag{I.6}$$

and at order ε^0 obtain from Eq. (I.4),

$$\mathcal{L}\varphi^{(0)} = 0. \tag{I.7}$$

The solution to (I.7) is $\varphi^{(0)}(T, \boldsymbol{r}, \boldsymbol{R}) = \Phi(T, \boldsymbol{R})$.

At order ε^1, Eq. (I.4) with the expansion (I.6) gives

$$\mathcal{L}\varphi^{(1)} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{R}} \Phi = 0. \tag{I.8}$$

We introduce the cell-average of a function f,

$$\langle f \rangle := \frac{1}{V} \int_\Omega f\, \mathrm{d}^3 r, \qquad V := \int_\Omega \mathrm{d}^3 r, \tag{I.9}$$

and cell-average Eq. (I.8), using $\left\langle \mathcal{L}\varphi^{(1)} \right\rangle = 0$, to obtain

$$\langle \boldsymbol{u} \rangle \cdot \nabla_{\boldsymbol{R}} \Phi = 0 \tag{I.10}$$

which is satisfied for $\langle \boldsymbol{u} \rangle = 0$.

From Eqs. (I.8) and (I.10) we must solve

$$\mathcal{L}\varphi^{(1)} + \boldsymbol{u}\cdot\nabla_{\boldsymbol{R}}\Phi = 0\,. \tag{I.11}$$

The solution to this is $\varphi^{(1)} = \boldsymbol{\chi}(\boldsymbol{r})\cdot\nabla_{\boldsymbol{R}}\Phi$, where

$$\mathcal{L}\boldsymbol{\chi} + \boldsymbol{u} = 0\,, \tag{I.12}$$

the so-called *cell problem*. Note that we must have $\langle\mathcal{L}\boldsymbol{\chi}\rangle = 0$ for the cell problem to have a solution, and that $\boldsymbol{\chi}$ is not unique since we can add a constant to it. Without loss of generality, choose $\langle\boldsymbol{\chi}\rangle = 0$.

Assuming the cell problem (I.12) has been solved, we can proceed to order ε^2 in Eq. (I.4),

$$\mathcal{L}\varphi^{(2)} + \partial_T\Phi + \boldsymbol{u}\cdot\nabla_{\boldsymbol{R}}\varphi^{(1)} = 2D\nabla_{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{R}}\varphi^{(1)} + D\Delta_{\boldsymbol{R}}\Phi\,. \tag{I.13}$$

Cell-averaging (I.13) and using $\left\langle\mathcal{L}\varphi^{(2)}\right\rangle = 0$, we find

$$\partial_T\Phi + \nabla_{\boldsymbol{R}}\cdot(\langle\boldsymbol{u}\,\boldsymbol{\chi}\rangle\cdot\nabla_{\boldsymbol{R}}\Phi) = 2D\nabla_{\boldsymbol{R}}\cdot(\langle\nabla_{\boldsymbol{r}}\boldsymbol{\chi}\rangle\cdot\nabla_{\boldsymbol{R}}\Phi) + D\Delta_{\boldsymbol{R}}\Phi\,. \tag{I.14}$$

The average $\langle\nabla_{\boldsymbol{r}}\boldsymbol{\chi}\rangle$ vanishes, and we thus finally obtain the homogenized diffusion equation

$$\partial_T\Phi = \nabla_{\boldsymbol{R}}\cdot(\mathbb{D}_{\text{eff}}\cdot\nabla_{\boldsymbol{R}}\Phi) \tag{I.15}$$

where the effective diffusivity tensor is

$$\mathbb{D}_{\text{eff}} \coloneqq D\,\mathbb{I} - \langle\boldsymbol{u}\,\boldsymbol{\chi}\rangle\ . \tag{I.16}$$

II. AN EXAMPLE

Consider the streamfunction for the *cellular flow*

$$\psi(x,y) = \sqrt{2}\,(U\ell/2\pi)\sin(2\pi x/\ell)\sin(2\pi y/\ell), \tag{II.1}$$

with velocity

$$\begin{aligned} u(x,y) &= \partial_y\psi = \sqrt{2}\,U\sin(2\pi x/\ell)\cos(2\pi y/\ell),\\ v(x,y) &= -\partial_x\psi = -\sqrt{2}\,U\cos(2\pi x/\ell)\sin(2\pi y/\ell). \end{aligned} \tag{II.2}$$

To compute the effective diffusivity, we need to solve the cell problem (I.12). Consider the ratio

$$\frac{|\boldsymbol{u}\cdot\nabla\boldsymbol{\chi}|}{|D\Delta\boldsymbol{\chi}|} \sim \frac{U\ell}{D} =: \text{Pe}, \tag{II.3}$$

where Pe is the *Péclet number*. If the Péclet number is small, we can neglect the advection term in the cell problem, and get the simplied equation $D\Delta\boldsymbol{\chi} = \boldsymbol{u}$, or

$$D\Delta\chi_x = \sqrt{2}\,U\sin(2\pi x/\ell)\cos(2\pi y/\ell), \qquad D\Delta\chi_y = -\sqrt{2}\,U\cos(2\pi x/\ell)\sin(2\pi y/\ell), \tag{II.4}$$

with solution

$$\boldsymbol{\chi} = -\frac{\ell^2}{9\pi^2 D}\,\boldsymbol{u}. \tag{II.5}$$

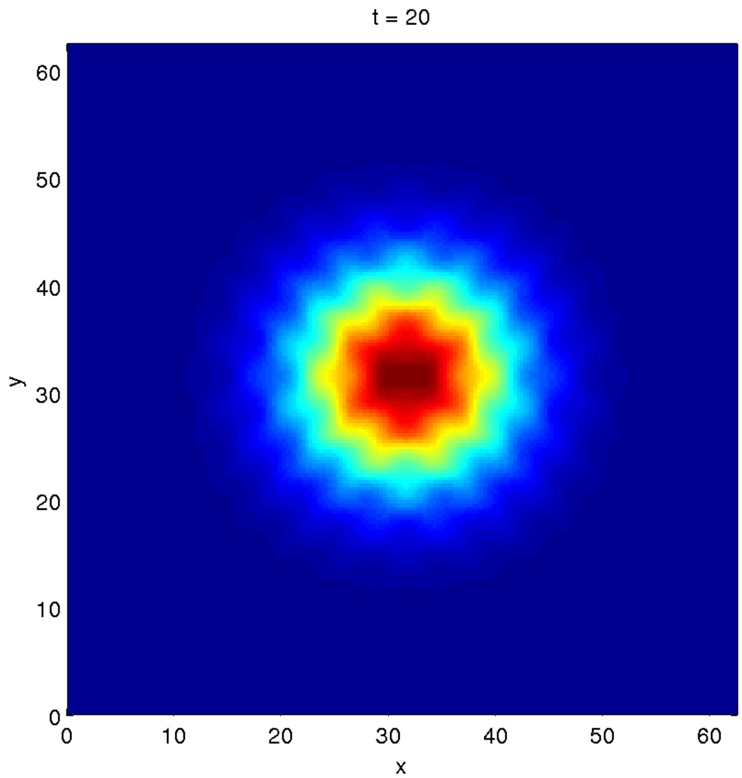

FIG. 1. Concentration field at $t = 20$ for $U = 1$, $\ell = 2\pi$, $D = 1$.

We can then easily compute the effective diffusivity tensor by using $\langle \boldsymbol{uu} \rangle = \frac{1}{2}U^2\mathbb{I}$ in (I.16):

$$\mathbb{D}_{\text{eff}} := D\left(1 + \frac{1}{16\pi^2}\,\text{Pe}^2\right)\mathbb{I}. \tag{II.6}$$

Figure 1 shows the concentration field for a numerical simulation at small Pe. In Figure 2 we compare the evolution of the variance to that implied by (II.6). Note that there is a short transient, since the initial condition has a small scale and so must spread out before scale separation is achieved.

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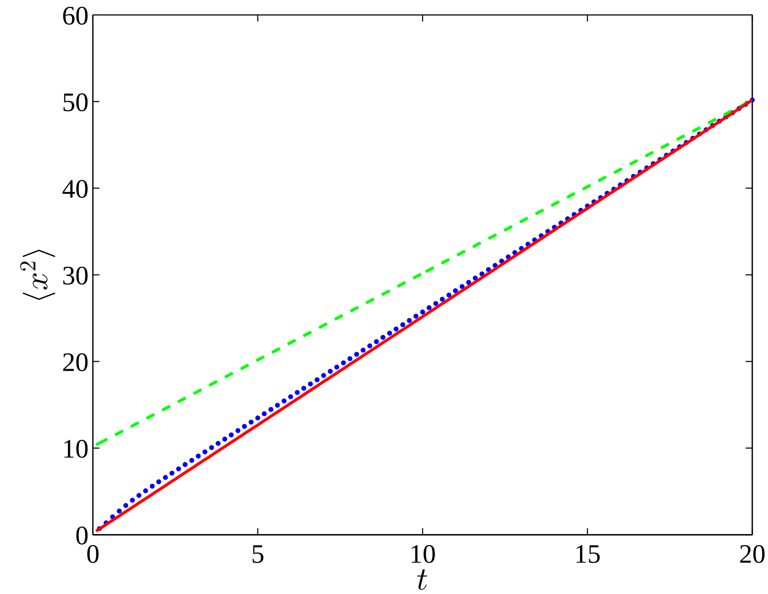

FIG. 2. Evolution of variance for $U = 1$, $\ell = 2\pi$, $D = 1$. The dots are numerical simulations, the green dashed line is $2Dt$, and the red line is $2\mathbb{D}_{\text{eff}}t$, where $\mathbb{D}_{\text{eff}}$ is defined in (II.6).

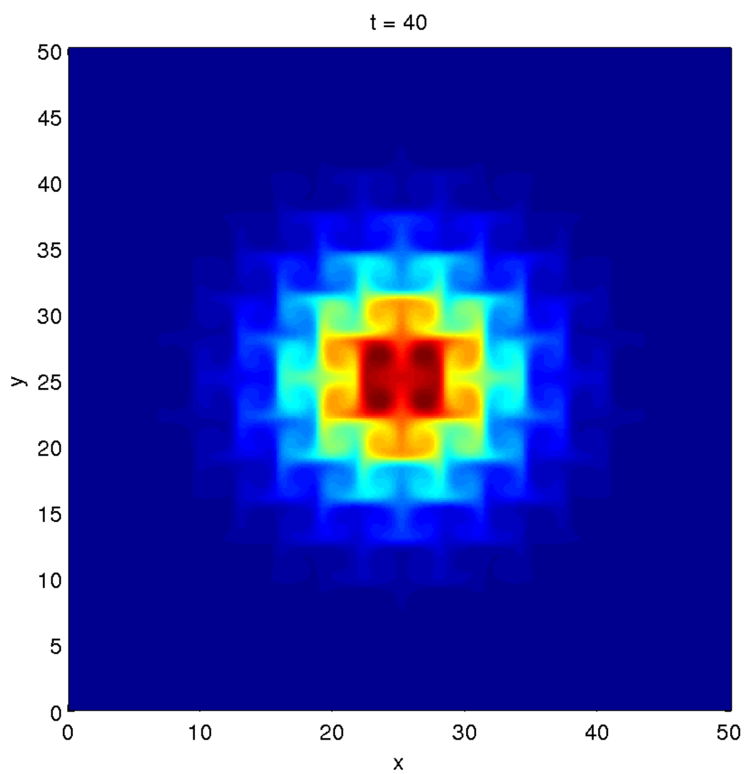

FIG. 3. Concentration field at $t = 40$ for $U = 1$, $\ell = 2\pi$, $D = 0.1$.

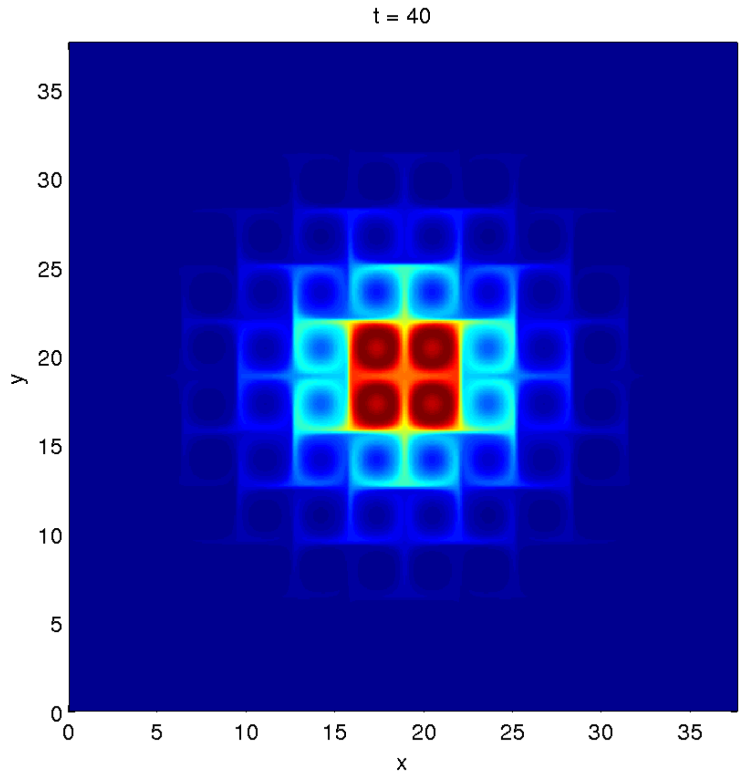

FIG. 4. Concentration field at $t = 40$ for $U = 1$, $\ell = 2\pi$, $D = 0.01$.

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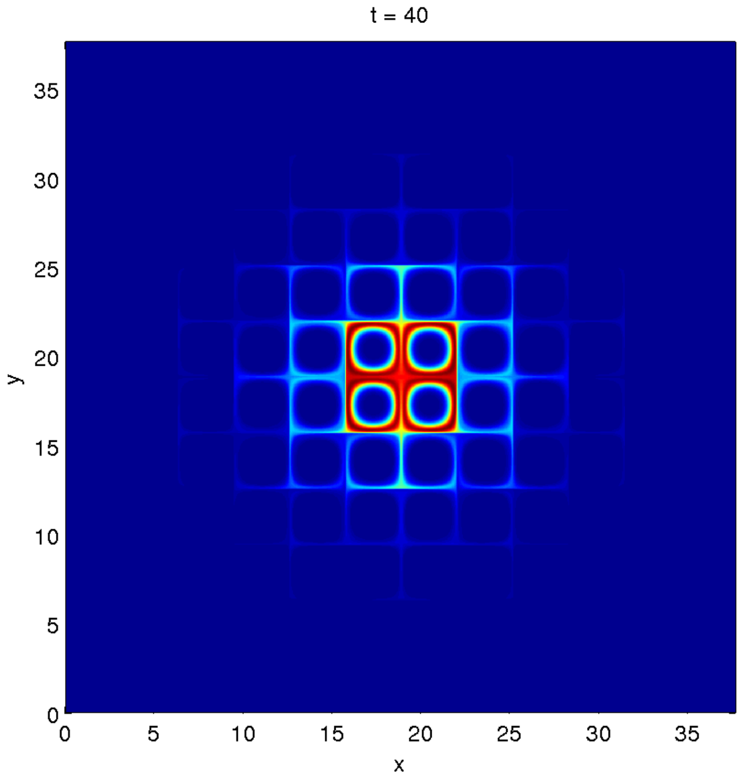

FIG. 5. Concentration field at $t = 40$ for $U = 1$, $\ell = 2\pi$, $D = 0.001$.

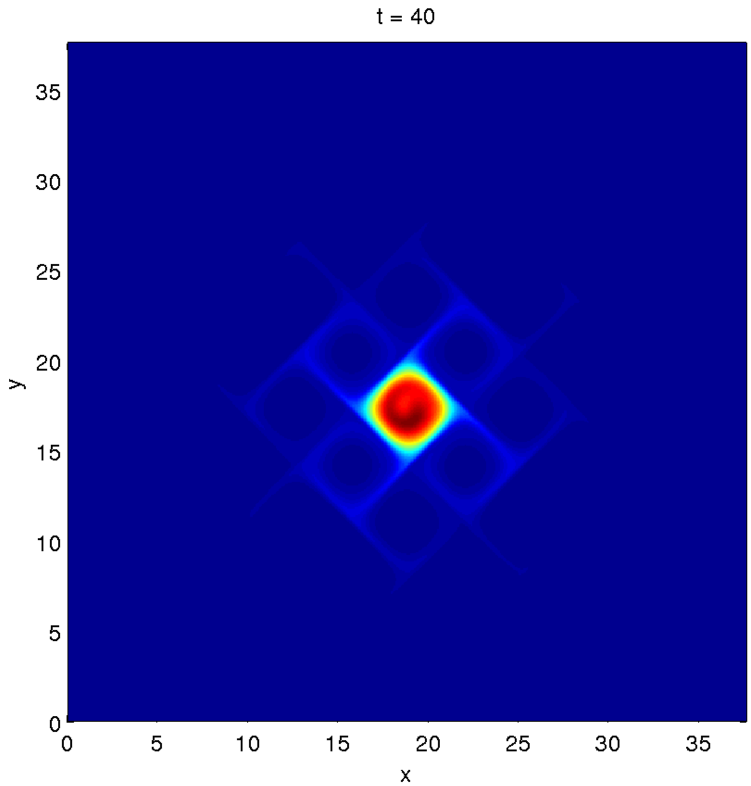

FIG. 6. Concentration field at $t = 40$ for the flow $\psi(x, y) = B \sin y + A \cos x$ with $D = 0.01$, and $B = -A = 1$. This flow has closed streamlines (see Crisanti *et al.*[1]).

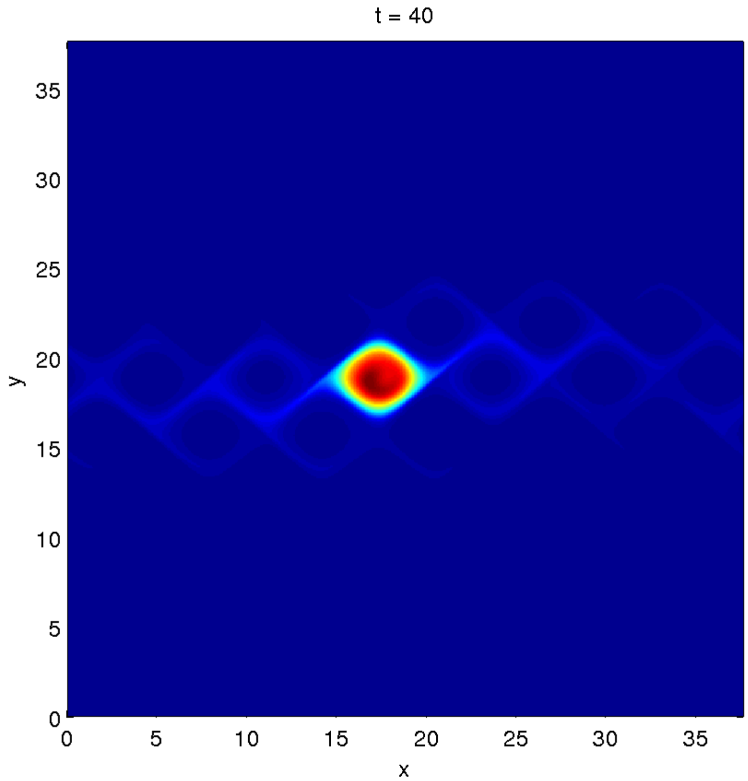

FIG. 7. Concentration field at $t = 40$ for the flow $\psi(x, y) = B\sin y + A\cos x$ with $D = 0.01$, and $B = 1$, $A = -1.3$. This flow has open streamlines (see Crisanti *et al.*[1]).