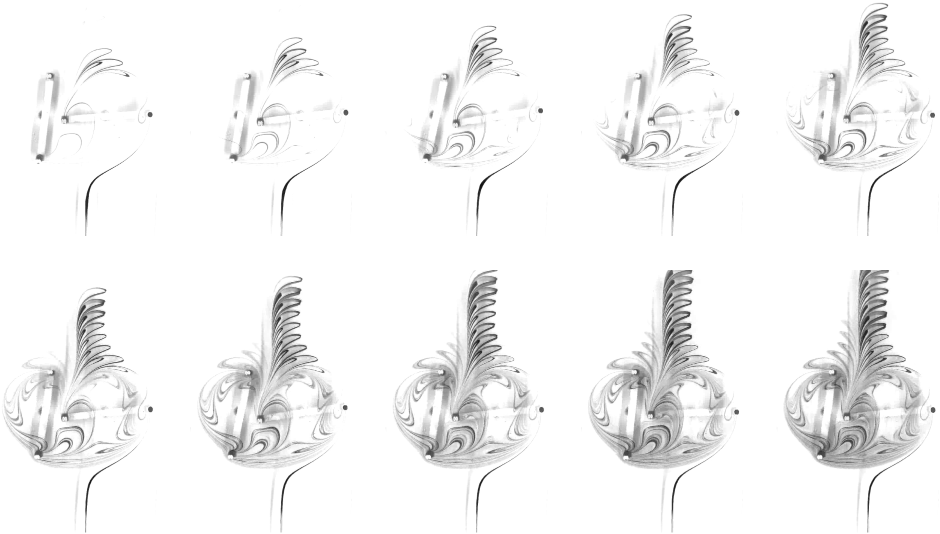
# Lecture 14: Strange eigenmodes and intermittency



play movie

# References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Pierrehumbert, R. T. (1994). *Chaos Solitons Fractals,* **4** (6), 1091–1110.

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# Lecture 14: Strange eigenmodes & intermittency

Following Vanneste (2006), consider

$$\underline{u} = \begin{pmatrix} 0 \\ V(x,t) \end{pmatrix}$$

and concentration $C(x,y,t)$.

$$\partial_t C + V C_y = \kappa \left( C_{xx} + C_{yy} \right)$$

Assume $2\pi$-periodic in $x$ and $y$. Let

$$C(x,y,t) = \mathrm{Re}\left[ e^{ily - \kappa l^2 t} \hat{C}(x,t) \right]$$

Then

$$\hat{C}_t + il\, V(x,t)\, \hat{C} = \kappa\, \hat{C}_{xx}$$

The velocity, being a function of $x$ only, does not couple $y$ modes

$$\hat{C}_t = f(\hat{C}, t)$$

Think of this as an $\infty$-dimensional dyn. sys

For long times, Oseledec says

$$\hat{C}(x,t) \sim D(x,t)\, e^{-\lambda t}, \quad t \to \infty$$

$\lim_{t\to\infty} \frac{\log D}{t} \to 0$ subexponential

Where $\lambda = -\lim_{t\to\infty} \frac{1}{t} \log \frac{\|\hat{C}(t)\|}{\|\hat{C}(0)\|}$ ← $L_2$

$-\lambda$ is the largest Lyapunov exponent for the $\hat{C}$ eq'n.

Finite time: $\lambda_t = -\frac{1}{t} \log \frac{\|\hat{C}(t)\|}{\|\hat{C}(0)\|}$

Moment decay rates:

$$\gamma_p = -\lim_{t\to\infty} \frac{1}{t} \log \frac{\langle \|\hat{C}\|^p \rangle (t)}{\langle \|\hat{C}\|^p \rangle (0)}$$

ensemble average

Recall also: $\gamma_p = \inf_{\lambda_t} \left[ p\lambda_t + g(\lambda_t) \right]$

$\lambda_t \to \lambda$ as $t \to \infty$

but $\gamma_p \neq p\lambda$ in general.

This is called <u>temporal intermittency</u>

Let's consider two models that show intermittency.

<u>type 1:</u> $\quad V(x,t) = f(t) \sin x$

with $\quad f(x) = \dfrac{\alpha}{l} \xi_n, \quad n \leq t < n+1$

$n = 0, 1, 2, \cdots$

$\xi_n$ are i.i.d. Gaussian (mean 0, unit var.)

This is a renewing flow.

With $\kappa = 0$, the equation

$$\hat{C}_t = -il V(x,t) \hat{C}_x$$

can be solved from $t = nt$ to $(n+1)t$ as

$$\hat{C}(x, n+1) = e^{-i\,\ell \underbrace{V(x,n)}_{\text{constant over interval}}} \hat{C}(x, n)$$

$$n \leq t < n+1$$

Let $\hat{C}_n = \hat{C}(x,n)$.

The solution with diffusion is tougher. It is more convenient to use pulsed diffusion:

$$\hat{C}_{n+1}(x) = e^{\kappa \partial_x^2} \left[ e^{-i\ell V(x,n)} \hat{C}(x,n) \right]$$

This is easy to integrate numerically: see

$$\boxed{\text{renflow('type1')}}$$ [Matlab]

for an example

Note that the realizations plotted all decay at roughly the same rate, but there are fluctuations.

The moments decay exponentially, but the rate $\to 0$ as $\kappa \to 0$.

Animation: $\boxed{\text{renflow('anim1')}}$

The animation clearly shows that $\hat{C}$ converges to a kind of "eigenmode", though the phase fluctuates.

These are often called generalized eigenmodes or eigenmodes in the sense of Oseledec.

Look for intermittency:

$$\boxed{\text{renflow\_gammap}}$$

Very weak intermittency (deviation from linear)

The peaks in $\hat{C}$ get narrower as $\kappa \to 0$.

Vanneste does a boundary-layer analysis to find

$$\lambda = \gamma_1 \sim 0.460\,(\kappa\alpha)^{2/3}$$

$$\gamma_2 \sim 0.881\,(\kappa\alpha)^{2/3}$$

Let's look a bit at what Vanneste did.

The decay is slow compared to the period.

Suggests modelling as stochastic equation with white noise:

Stratonovich $\quad$ ($W_t$ = Wiener process)

$$\hat{C}_t + i\alpha \sin x \, \hat{C} \circ \dot{W}_t = \kappa \hat{C}_{xx}$$

Rescale: $x = \frac{\pi}{2} + \kappa^{1/6} \alpha^{-1/3} X$, $\quad t = (\kappa\alpha)^{-2/3} T$

$$\hat{C}(x,t) = e^{-i\alpha W_t} \tilde{C}(X,T)$$

$$\hat{C}_t = e^{-i\alpha W_t}\left(-i\alpha \tilde{C} \circ \dot{W}_t + \tilde{C}_T (\kappa\alpha)^{2/3}\right)$$

$$\hat{C}_{xx} = e^{-i\alpha W_t} \tilde{C}_{XX} \left(\kappa^{-1/6} \alpha^{1/3}\right)^2$$

$$-i\alpha \dot{W}_t \tilde{C} + \tilde{C}_T (\kappa\alpha)^{2/3} + i\alpha \sin\left(\frac{\pi}{2} + \kappa^{1/6} \alpha^{-1/3} X\right) \tilde{C} \circ \dot{W}_t$$
$$= \left(\kappa^{2/3} \alpha^{2/3}\right) \tilde{C}_{XX}$$

Use $\sin\left(\frac{\pi}{2} + \varepsilon\right) = 1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4)$:

$$\cancel{i\alpha \tilde{C} \circ \dot{W}_t} + (\kappa\alpha)^{2/3} \tilde{C}_T +$$
$$i\alpha \left(\cancel{1} - \kappa^{1/3} \alpha^{-2/3} \frac{X^2}{2}\right) \tilde{C} \circ \dot{W}_t = (\kappa\alpha)^{2/3} \tilde{C}_{XX}$$

$$\tilde{C}_T - \frac{i}{2}(\kappa\alpha)^{-1/3} X^2 \tilde{C} \circ \dot{W}_t = \tilde{C}_{XX}$$

But note that $t = (\kappa\alpha)^{-2/3} T$,

$$\sqrt{t} = (\kappa\alpha)^{-1/3} \sqrt{T}$$

$W_t$ satisfies a "Brownian scaling":

$$W_t = \frac{1}{\sqrt{c}} W_{ct}$$

So if $c = (\kappa\alpha)^{-2/3}$, $W_t = (\kappa\alpha)^{1/3} W_T$,
and hence

$$\tilde{C}_T - \frac{1}{2} i X^2 \tilde{C} \circ \tilde{W}_T = \tilde{C}_{XX}.$$

This shows that the width of the boundary layer scales as $\kappa^{1/6}\alpha^{2/3}$.

It also gives us the time scale $(\kappa\alpha)^{-2/3}$.

$$\tilde{C}(X,T) = e^{-(a(T)X^2 + b(T))}$$

$$\tilde{C}_X = -2Xa(T)\tilde{C}$$

$$\tilde{C}_{XX} = \left(-2a(T) + 4X^2 a^2(T)\right)\tilde{C}$$

$$\tilde{C}_T = \tilde{C}\left(-\dot{a}X^2 - \dot{b}\right)$$

$$-\dot{a}X^2 - \dot{b} - \frac{i}{2}X^2\dot{W}_T$$

$$= -2a + 4a^2X^2$$

Hence: $\left.\begin{aligned} \dot{a} &= -4a^2 - \frac{i}{2}\dot{W}_T \\ \dot{b} &= 2a \end{aligned}\right\}$ coupled SDEs

Now consider model <u>type 2</u>:

$$V(x,t) = \frac{\alpha}{l} \sin\left(x + \phi(t)\right)$$

$$\phi(t) = \phi_n \in [0, 2\pi), \quad n \leq t < n+1$$

$\boxed{\text{renflow('type2')}}$ $\boxed{\text{renflow('anim2')}}$

This exhibits a lot more intermittency (see movies, figures at the end).

$$\sin(x + \phi(t)) = \sin x \cos\phi + \cos x \sin\phi$$

Note that $E(\cos\phi) = E(\sin\phi) = 0$

$$E(\sin^2\phi) = E(\cos^2\phi) = \frac{1}{2}, \quad E(\sin\phi\cos\phi) = 0$$

So approximate $\sin(x+\phi) \simeq \frac{\dot{W}_t^1}{\sqrt{2}} \sin x + \frac{\dot{W}_t^2}{\sqrt{2}} \cos x$

$\dot{W}_t^1$, $\dot{W}_t^2$ independent Wiener processes.

Hence

$$\hat{C}_t + \frac{i\alpha}{\sqrt{2}} \hat{C} \circ \left( \dot{W}_t^1 \sin x + \dot{W}_t^2 \cos x \right) = \kappa \hat{C}_{xx}$$

Let $\hat{C}(x,t) = \rho(x,t)\, e^{i\theta(x,t)}$ $\quad \rho \in \mathbb{R}$

$$\hat{C}_t = (\rho_t + i\theta\rho)\,\hat{C}$$

$$\hat{C}_x = (\rho_x + i\rho\theta_x)\,\hat{C}$$

$$\hat{C}_{xx} = (\rho_{xx} + 2i\rho_x\theta_x + i\rho\theta_{xx} - \rho\theta_x^2)\,\hat{C}$$

$$\Rightarrow \rho_t = \kappa\rho_{xx} - \kappa\rho\theta_x^2$$

$$\theta_t = -\frac{\alpha}{\sqrt{2}} \left( \dot{W}_t^1 \sin x + \dot{W}_t^2 \cos x \right) + \kappa \left( \theta_{xx} + \frac{2\rho_x\theta_x}{\rho} \right)$$

For short time, diffusion can be neglected.

The phase is then given by

$$\theta(x,t) = -\frac{\alpha}{\sqrt{2}}\left(W_t^1 \sin x + W_t^2 \cos x\right)$$

$$(\theta(x,0)=0)$$

Hence, $E\theta^2 \sim t$ The phase "diffuses"

Note that $\rho_t = \underbrace{\kappa \rho_{xx}}_{\text{neglect for now}} - \kappa \rho \underline{\underline{\theta_x^2}}$ $\rho$ = constant for $\kappa = 0$

$$\rho_t \approx -\kappa \rho \theta_x^2 \Rightarrow \rho(x,t) \approx \rho_0(x) e^{-\kappa \int_0^t \theta_x^2 dt}$$

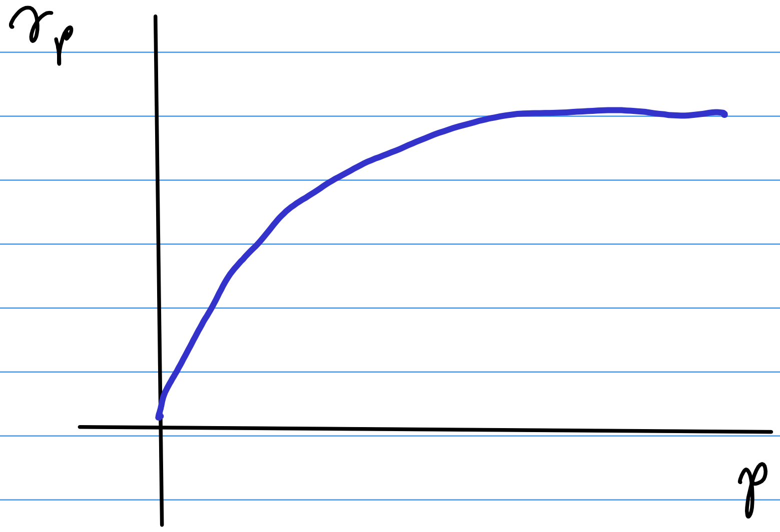
This describes the early stages of the evolution.

$$\rho \sim e^{-\kappa t^{\boxed{2}}}$$ $\leftarrow$ not an eigenmode

Vanneste shows in Appendix B that

$$E\rho^p(x,t) = \frac{\rho_0^p(x)}{\cosh^{1/2}\left[\alpha(\kappa p)^{1/2}\right]}$$

$$\sim \rho_0^p(x) e^{-\alpha(\kappa p)^{1/2} t/2}, \quad t \gg 1$$




Note that this agrees reasonably well with the numerical sim, even though we used an "early time" approx.

This is because $\gamma_\rho$ is dominated by the rare realizations with low stretching, which are in the "early time" regime.

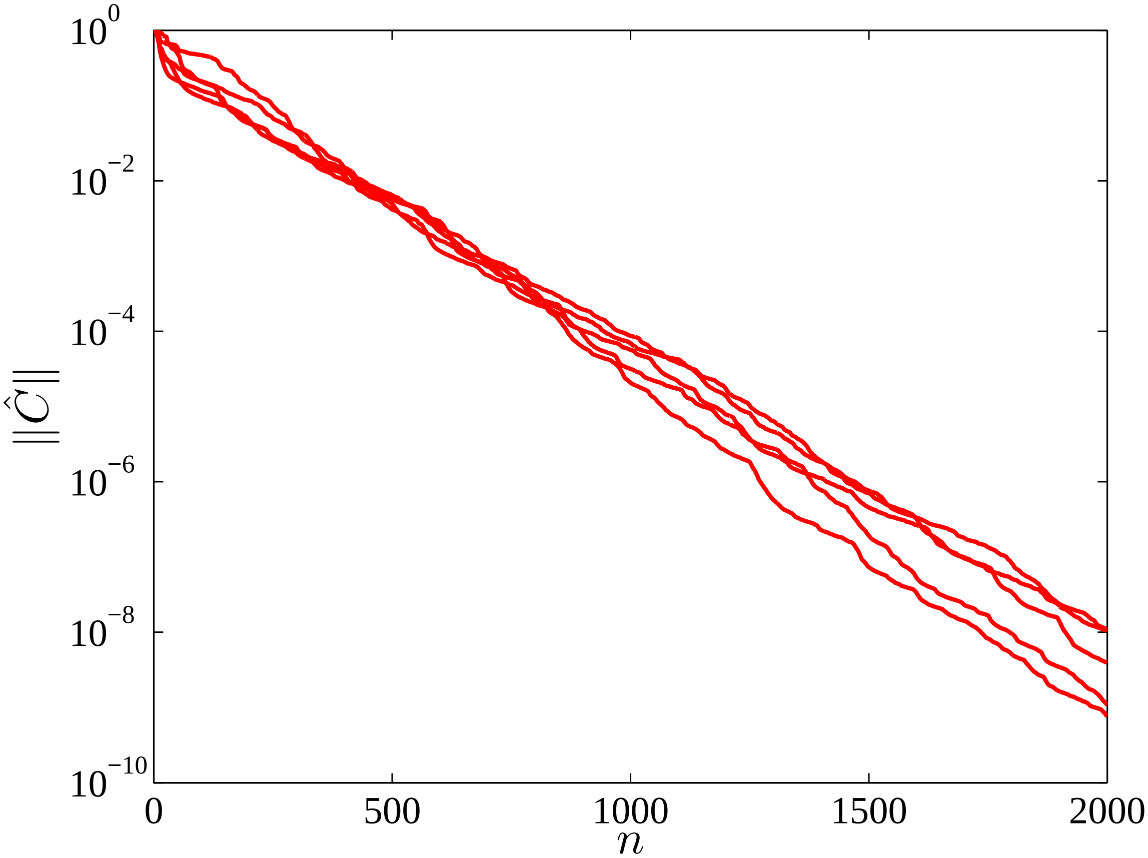



Figure 1: Decay of the concentration $L_2$ norm for the Type 1 flow (random amplitude) in J. Vanneste, *Intermittency of passive-scalar decay: Strange eigenmodes in random shear flows*, Phys. Fluids, 18 (2006), p. 087108, for five realizations.

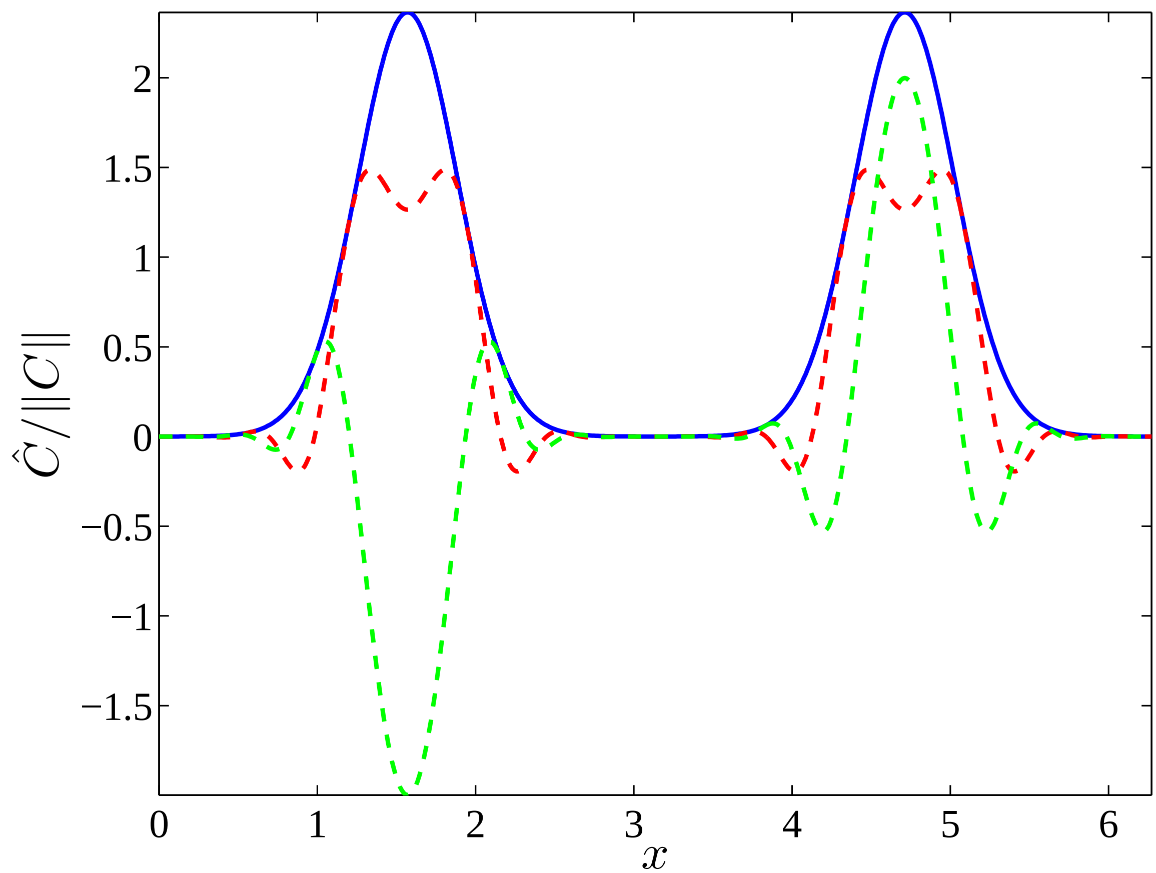



Figure 2: Generalized eigenmode of the Type 1 flow (random amplitude) in J. Vanneste, *Intermittency of passive-scalar decay: Strange eigenmodes in random shear flows*, Phys. Fluids, 18 (2006), p. 087108.

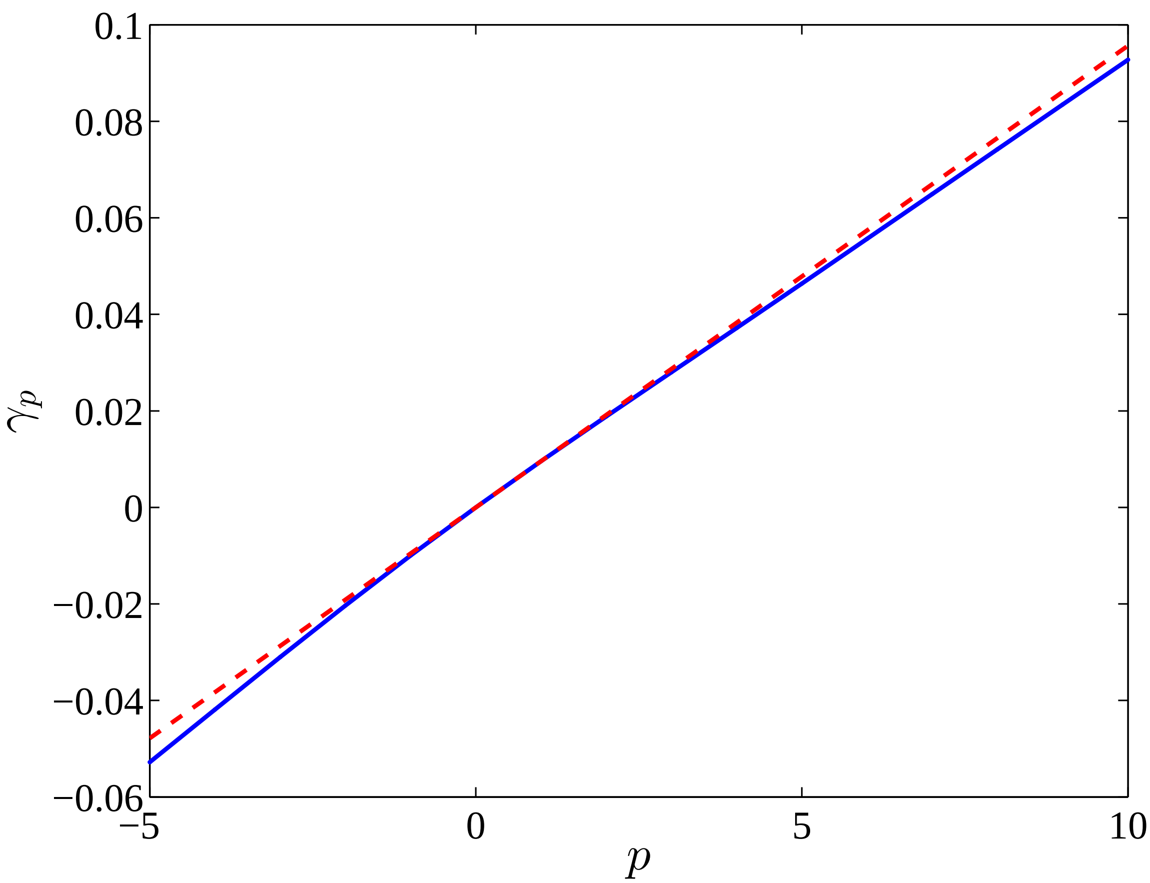



Figure 3: Weak intermittency of the Type 1 flow (random amplitude) in J. Vanneste, *Intermittency of passive-scalar decay: Strange eigenmodes in random shear flows*, Phys. Fluids, 18 (2006), p. 087108. The dashed line is the linear scaling.

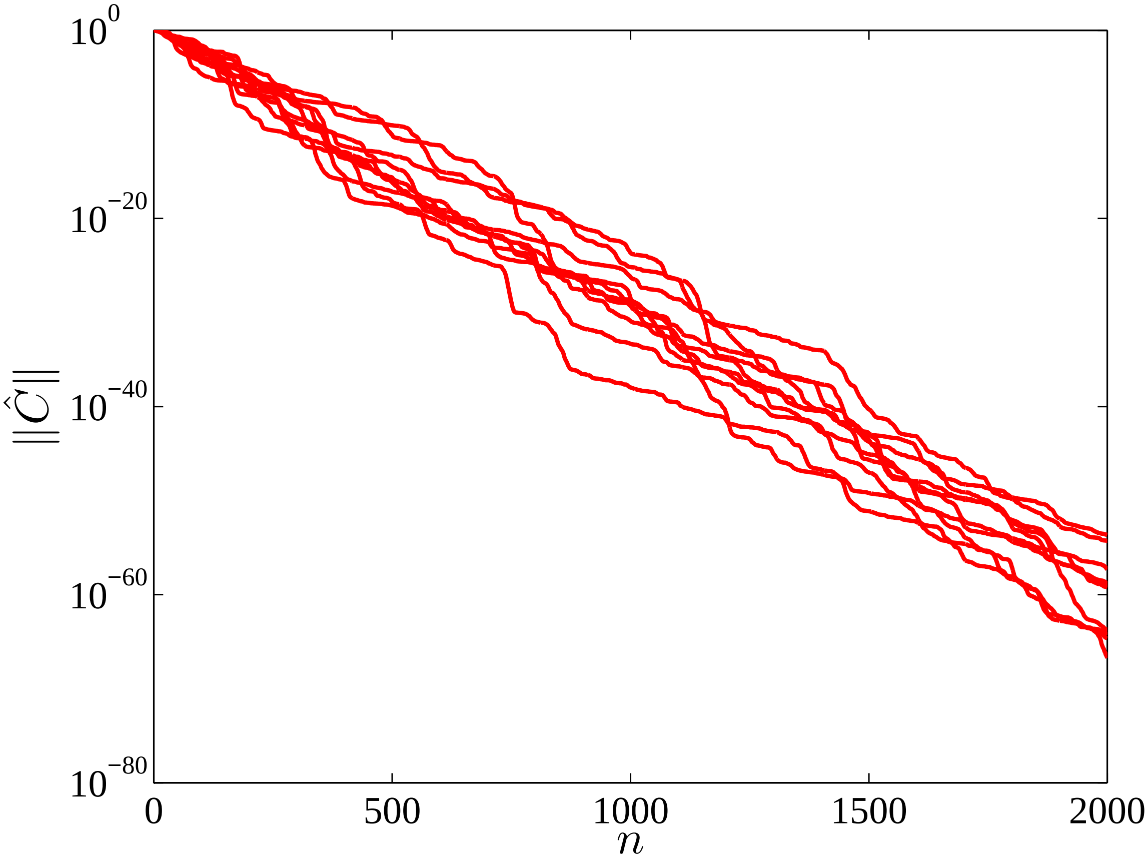



Figure 4: Decay of the concentration $L_2$ norm for the Type 2 flow (random phase) in J. Vanneste, *Intermittency of passive-scalar decay: Strange eigenmodes in random shear flows*, Phys. Fluids, 18 (2006), p. 087108, for five realizations.

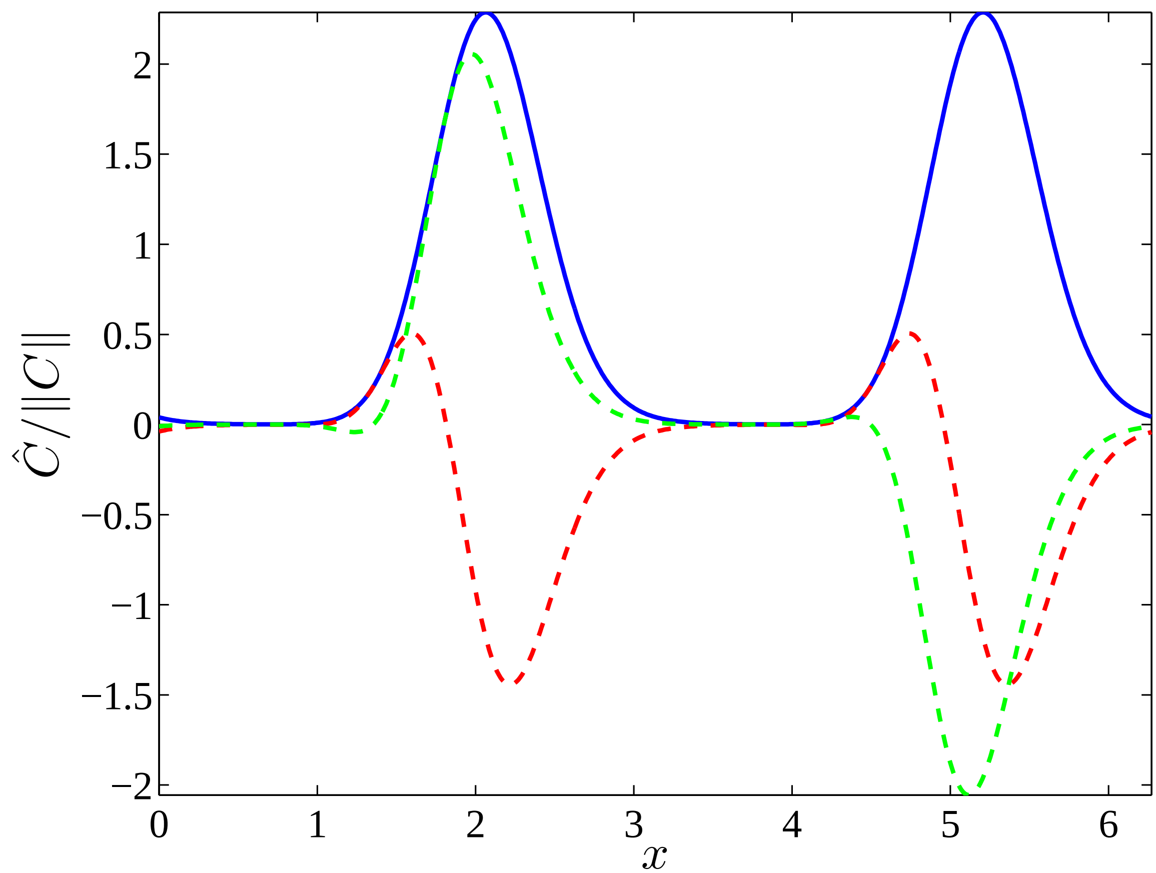



Figure 5: Generalized eigenmode of the Type 2 flow (random phase) in J. Vanneste, *Intermittency of passive-scalar decay: Strange eigenmodes in random shear flows*, Phys. Fluids, 18 (2006), p. 087108.

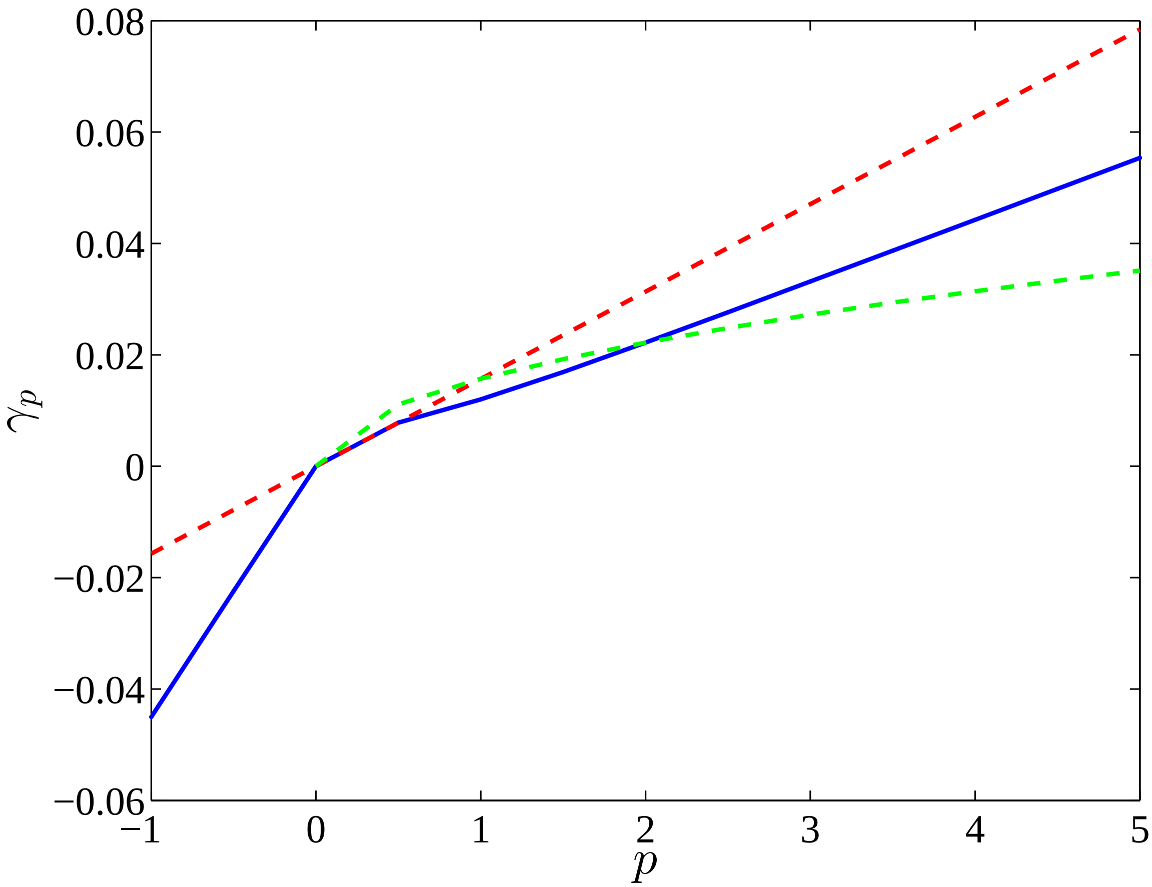



Figure 6: Weak intermittency of the Type 2 flow(random phase) in J. VANNESTE, *Intermittency of passive-scalar decay: Strange eigenmodes in random shear flows*, Phys. Fluids, 18 (2006), p. 087108. The dashed line is the linear scaling.

```
function Cnorm = renflow

type = 1; % the two flows in Vanneste: type 1 or 2
Nstep = 2000; Nreal = 100; N = 512;
kappa = 1e-3; alpha = pi; p = -1:2;

kmin = floor(-(N-1)/2); kmax = floor((N-1)/2); k = [0:kmax kmin:-1];
x = linspace(0,2*pi,N+1); x = x(1:end-1);

switch type
 case 1
  advec = @() exp(-i*alpha*randn(Nreal,1)*sin(x));
 case 2
  advec = @() exp(-i*alpha*sin(tensorsum(x,2*pi*rand(Nreal,1))));
end

Cnorm = zeros(Nreal,Nstep+1,length(p)); C = ones(Nreal,N);
Cnorm(:,1,:) = Cnorms(C,p);
diff = diag(sparse(exp(-kappa*k.^2)));
for n = 1:Nstep
  C = advec().*C;                        % advection step
  Ck = fft(C,[],2);                      % Fourier transform
  Ck = Ck*diff;                          % diffusion step
  C = ifft(Ck,[],2);                     % inverse Fourier transform
  Cnorm(:,n+1,:) = Cnorms(C,p);
end
Cnorm = squeeze(mean(Cnorm)); % average over realizations

%======================================================================
function Cp = Cnorms(C,p)

Cp = zeros(size(C,1),length(p));
for ip = 1:length(p)
  Cp(:,ip) = sqrt(sum(C.*conj(C),2)/size(C,2)).^p(ip);
end
```

Figure 7: A simplified version of the Matlab code `renflow.m`, which implements the evolution of a passive scalar stirred by the two model flows in J. Vanneste, *Intermittency of passive-scalar decay: Strange eigenmodes in random shear flows*, Phys. Fluids, 18 (2006), p. 087108.