Lecture 13: Rate of decay and local stretching

Recall from many lectures ago that for a 2D, extensional, incompressible flow, the concentration of a passive scalar:

$$\theta(x, y, t) \sim e^{-\lambda t}$$

λ = rate of strain

with a Gaussian cross-section.

Now imagine the blow is being subjected to a random renewing flow.

Assume that τ (the correlation time) is large enough that our Gaussian blob aligns rapidly:

$$\lambda \tau \gg 1$$

Then at each application the intensity of θ decays by a factor $e^{h\tau}$

Consider $\langle |\theta|^p \rangle$, where the expected value is over the random matrices in our flow.

Then:

$$\langle |\theta|^p \rangle \sim \int e^{-ph\tau N} \, e^{-N g(h)} \, dh$$

time $= N\tau$ Cramér function

$$\sim \int e^{-N(ph + g(h))} \, dh$$

For large N, the rate of decay is thus

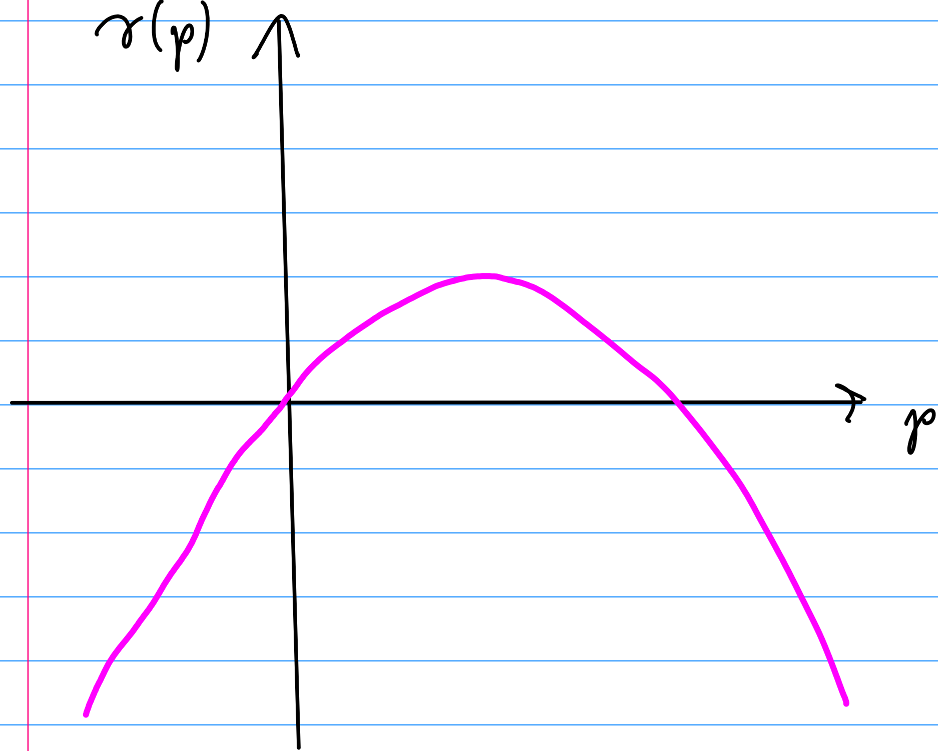
$$\gamma(p) = \frac{1}{\tau} \inf_h \left(ph + g(h) \right)$$

$$\langle |\theta|^p \rangle \sim e^{-\gamma(p) t}, \quad t \to \infty$$

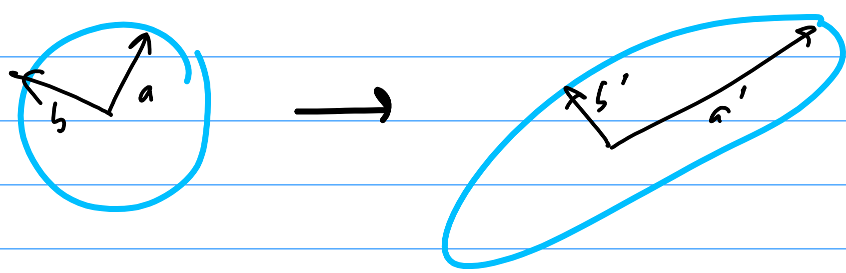
Thus,

$$\gamma(p) = -\ell(-p)$$

generalized Lyapunov exponent



There is one crucial difference: we cannot consider negative h. Why? h here is the stretching of the longest axis of the ellipsoid:



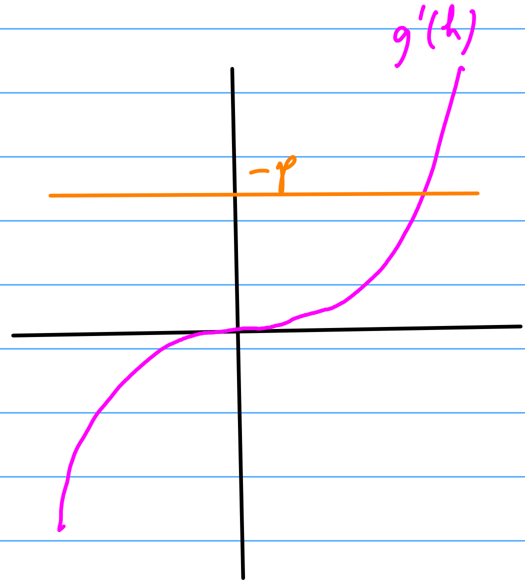
This is a nondecreasing quantity, unlike the growth of a line segment ℓ.

So consider: $\gamma(p) = \inf_{h>0} (ph + g(h))$

convex

saddle point

$$p + g'(h_*(p)) = 0$$



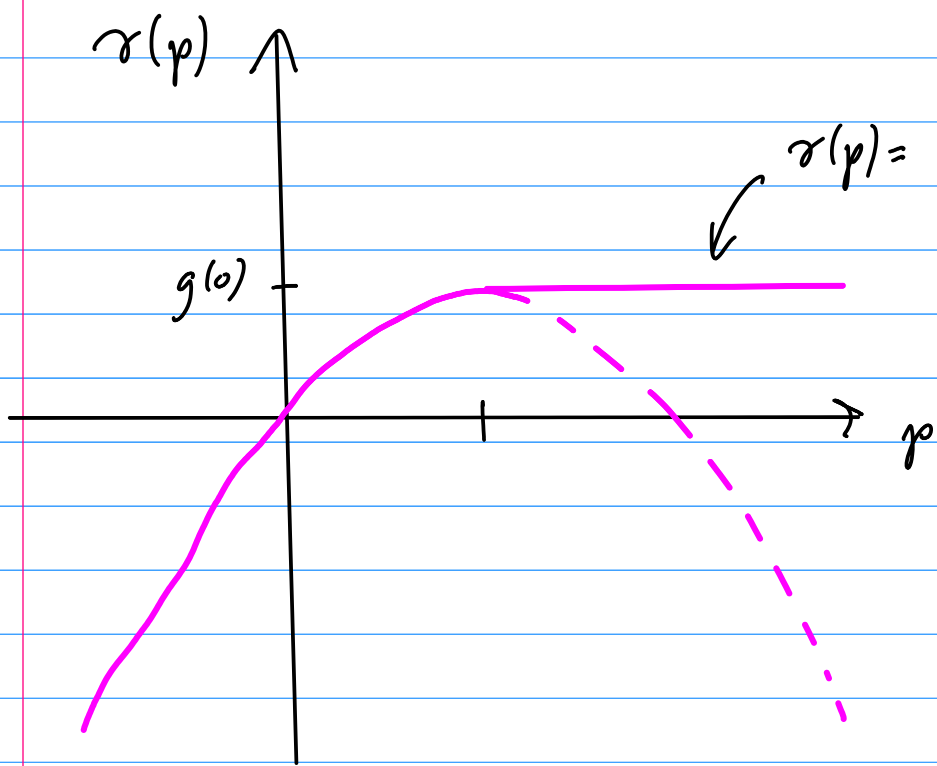
$$\gamma(p) = p h_*(p) + g(h_*(p))$$

$$\gamma'(p) = h_*(p) + g'(h_*(p))\, h_*'(p) + p h_*'(p)$$
$$= h_*(p)$$

So $\gamma'(p) = 0$ when $h_*(p) = 0 \Rightarrow h = 0$.

So the point were the saddle point is $h=0$ coincides with the extremum of $\ell(p)$!

We must amend our figure for $\gamma(p)$ to reflect this:



Thus, $$\gamma(p) = \begin{cases} \inf\limits_h \left(ph + g(h)\right) & , \; h_*(p) > 0 \\ g(0) & , \; h_*(p) < 0 \end{cases}$$

The decay of $\langle |\theta|^p \rangle$ for large p is thus completely dominated by realizations with zero stretching.

→ becomes independent of p.