Lecture 12: Generalized Lyapunov exponents

Recall from last time, for a linear renewing flow:

$$2\langle\log\|\boldsymbol{\ell}'\|\rangle = \log a + \frac{1}{\pi}\int_0^\pi \log(1+\alpha\,\cos\psi)\,\mathrm{d}\psi. \tag{1}$$

with

$$a = \cosh^2\zeta + (\Gamma^2+\Omega^2)\sinh^2\zeta, \qquad \alpha^2 = 1-(\Gamma^2\cosh 2\zeta - \Omega^2)^{-2}, \qquad 0\le\alpha<1. \tag{2}$$

More generally, consider the growth rate of $\|\boldsymbol{\ell}\|^q$, which is obtained by computing

$$\langle\|\boldsymbol{\ell}'\|^q\rangle = \frac{1}{\pi}\int_0^\pi (a + b\,\sin 2\theta + c\,\cos 2\theta)^{q/2}\,\mathrm{d}\theta\,. \tag{3}$$

Using the same method as before,

$$\langle\|\boldsymbol{\ell}'\|^q\rangle = a^{q/2}\,\frac{1}{\pi}\int_0^\pi (1+\alpha\,\cos\psi)^{q/2}\,\mathrm{d}\psi. \tag{4}$$

The integral can be evaluated in terms of a hypergeometric function,

$$\langle\|\boldsymbol{\ell}'\|^q\rangle = a^{q/2}\,(1-\alpha)^{q/2}\,{}_2F_1\left(\tfrac12,-\tfrac{q}{2};1;-\tfrac{2\alpha}{1-\alpha}\right). \tag{5}$$

We then define *generalized Lyapunov exponents* as

$$\ell(q) := \frac{1}{\tau}\log\langle\|\boldsymbol{\ell}'\|^q\rangle = \frac{1}{\tau}\log\left(\left(\tfrac{1-\alpha}{1+\alpha}\right)^{q/4}\,{}_2F_1\left(\tfrac12,-\tfrac{q}{2};1;-\tfrac{2\alpha}{1-\alpha}\right)\right). \tag{6}$$

This is plotted in Fig. 1: observe that $\ell(0)=\ell(-2)=0$, the curve has a minimum at $q=-1$, and it is symmetric about that value. These features all follow from the incompressibility of the flow, as we'll explain below.

For large $q>0$, we can use the saddle point method to carry out the integral (4):

$$\begin{aligned}
\langle\|\boldsymbol{\ell}'\|^q\rangle &= a^{q/2}\,\frac{1}{\pi}\int_0^\pi \exp(\tfrac12 q\log(1+\alpha\,\cos\psi))\,\mathrm{d}\psi \\
&\simeq a^{q/2}\,\frac{1}{\pi}\int_0^\infty \exp(\tfrac12 q\log(1+\alpha-\tfrac12\alpha\psi^2))\,\mathrm{d}\psi \\
&= a^{q/2}\,(1+\alpha)^{q/2}\,\frac{1}{\pi}\int_0^\infty \exp\left(-\tfrac14 q\tfrac{\alpha}{1+\alpha}\,\psi^2\right)\,\mathrm{d}\psi \\
&= (a(1+\alpha))^{q/2}\sqrt{\frac{1+\alpha}{\pi\alpha q}},
\end{aligned}$$

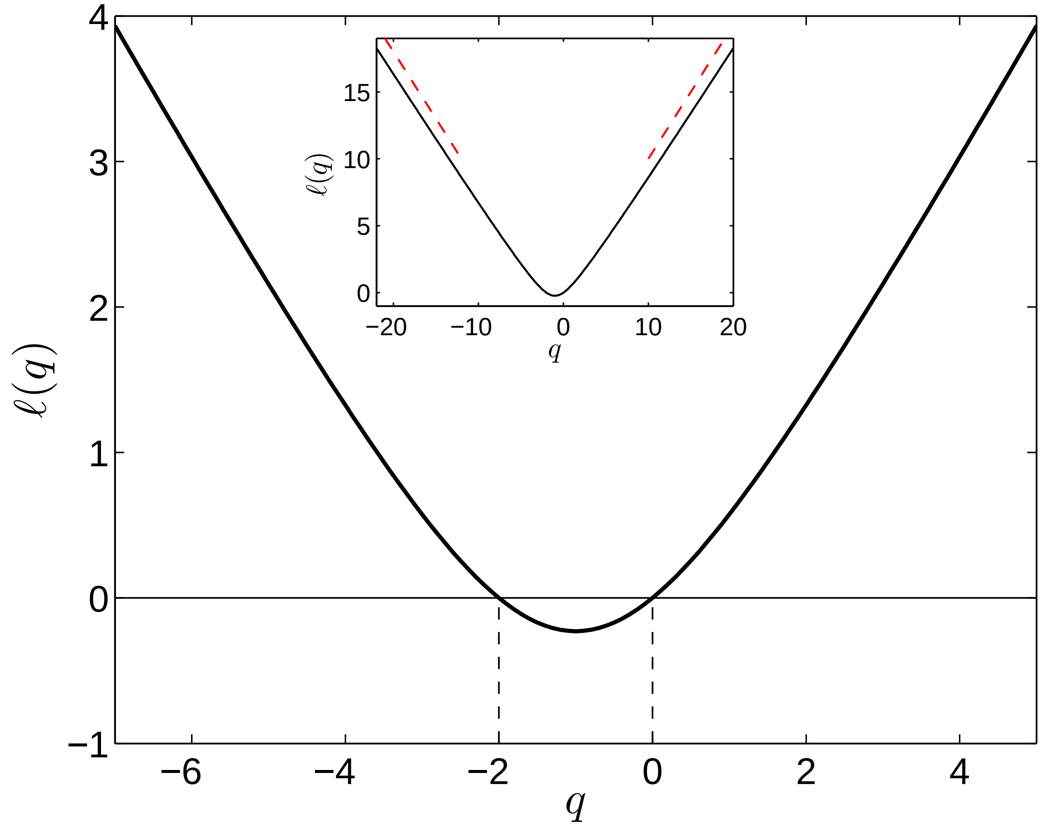

Figure 1: Generalized Lyapunov exponents $\ell(q)$ from (6) for $\gamma = 1$, $\omega = 0$, $\tau = 1$. The inset shows the large-$|q|$ asymptotes $\frac{1}{4\tau}|q|\log((1+\alpha)/(1-\alpha))$.

so that

$$\ell(q) = \tfrac{1}{2\tau}\, q \log(a(1+\alpha)) - \tfrac{1}{2\tau} \log q + \tfrac{1}{2\tau} \log\left(\tfrac{1+\alpha}{\pi\alpha}\right) + \mathrm{O}\big(q^{-1}\big)\,, \qquad q \gg 1. \tag{7}$$

We can do the same for $q < 0$, $|q| \gg 1$; the saddle point is then at the minimum $\psi = \pi$, and we find the leading order form $\ell(q) \sim \frac{1}{4\tau}|q| \log((1+\alpha)/(1-\alpha))$, which is shown in an inset to Fig. 1.

The 'true' Lyapunov exponent is $\lambda = \ell\,'(0)$:

$$\ell'(0) = \frac{1}{\tau}\, \frac{\langle \|\boldsymbol{\ell}'\|^q \log\|\boldsymbol{\ell}'\| \rangle}{\langle \|\boldsymbol{\ell}'\|^q \rangle}\bigg|_{q=0} = \frac{1}{\tau} \langle \log\|\boldsymbol{\ell}'\| \rangle. \tag{8}$$

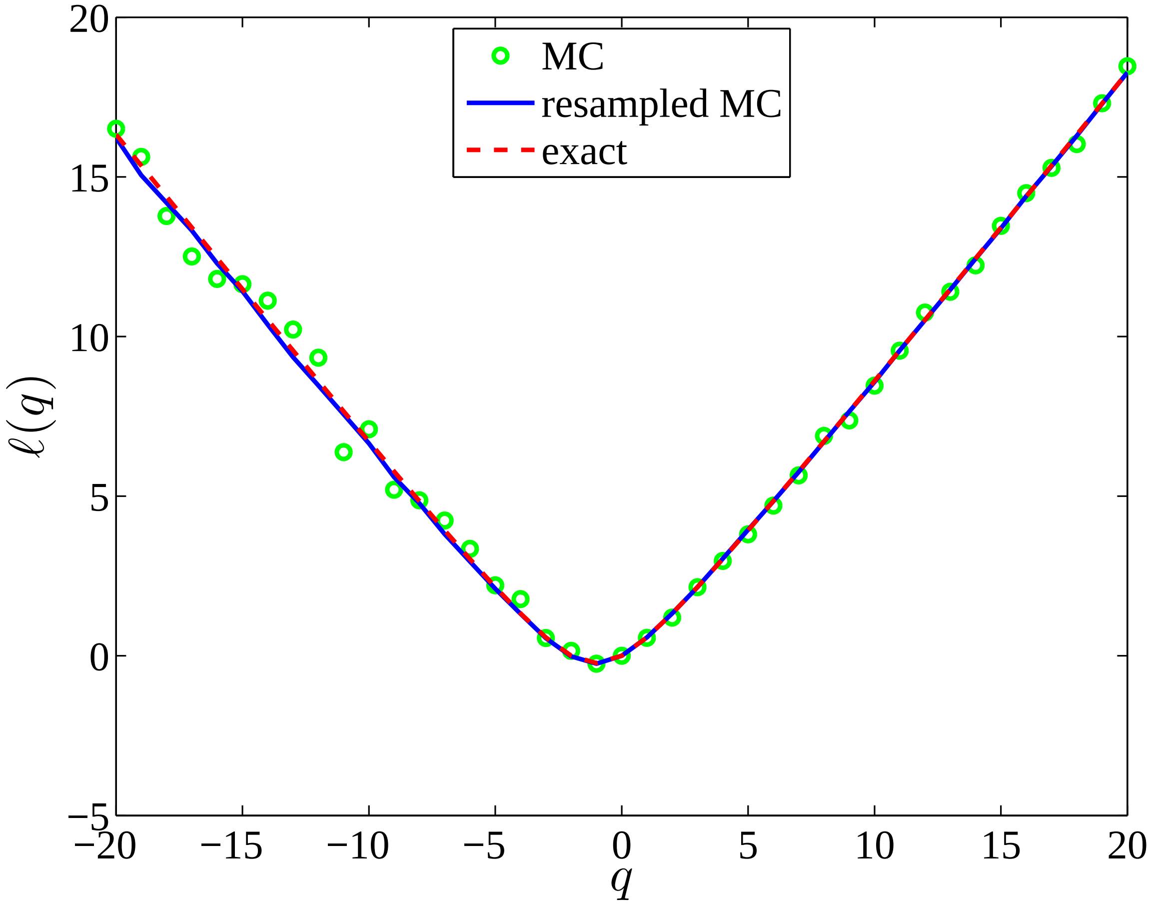

Figure 2: A comparison of resampled Monte–Carlo (J. Vanneste, *Estimating generalized Lyapunov exponents for products of random matrices*, Phys. Rev. E, 81 (2010), p. 036701) and direct Monte–Carlo for the linear renewing flow, with $\tau = 1$, $\gamma = 1$, $\omega = 0$, $K = 100$, and $N = 100$. The resampled MC is far better than plain MC, especially for negative q. See Fig. 3 for the Matlab code used to generate this figure.

```
% Resampled Monte-Carlo to compute the generalized Lyapunov exponents l(q)
% See J. Vanneste, Phys. Rev. E 81, 036701 (2010).
K = 100; N = 100; gamma = 1; omega = 0; q = -20:20;
zeta = sqrt(gamma^2-omega^2);
Gamma = gamma/zeta; Omega = omega/zeta;
cz = cosh(zeta); sz = sinh(zeta);
rng('default');
ev = zeros(K,2); ev(:,1) = 1;  % initial vectors [1 0]
evh = zeros(size(ev)); normevh = zeros(1,K); beta = zeros(1,N);
ell = []; ell0 = []; ellth = []; A = zeros(K,2,2);
for iq = 1:length(q)
  for n = 1:N
    % random matrix
    th2 = 2*pi*rand(1,K); ct = cos(th2); st = sin(th2);
    A(:,1,1) = cz + Gamma*ct*sz;      A(:,1,2) = (Gamma*st - Omega)*sz;
    A(:,2,1) = (Gamma*st + Omega)*sz; A(:,2,2) = cz - Gamma*ct*sz;
    % multiply matrix ev by matrix A (vectorized over realisations K)
    evh(:,1) = A(:,1,1).*ev(:,1) + A(:,1,2).*ev(:,2);
    evh(:,2) = A(:,2,1).*ev(:,1) + A(:,2,2).*ev(:,2);
    % make unit vector evh
    normevh = sqrt(evh(:,1).^2 + evh(:,2).^2);
    evh(:,1) = evh(:,1)./normevh; evh(:,2) = evh(:,2)./normevh;
    % save q^th power of the norm
    alpha = normevh.^q(iq);
    % resampling
    gamma = cumsum(alpha); beta(n) = gamma(K);
    eps = beta(n)*rand(1,K);
    for k = 1:K
      ii = find(gamma-eps(k) >= 0);
      ev(k,:) = evh(ii(1),:);
    end
  end
  % l(q) without resampling
  ell0 = [ell0 log(mean(alpha))];
  % l(q) with resampling
  ell = [ell mean(log(beta))-log(K)];
  % the analytic expression for l(q)
  aa = sqrt(1 - (Gamma^2*cosh(2*zeta) - Omega^2)^-2);
  ellth = [ellth log(((1-aa)/(1+aa))^(q(iq)/4)*...
     hypergeom([.5 -q(iq)/2],1,-2*aa/(1-aa)))];
end

plot(q,ell0,'go','LineWidth',2), hold on
plot(q,ell,'-','LineWidth',2)
plot(q,ellth,'r--','LineWidth',2)
legend('MC','resampled MC','exact','Location','North')
xlabel('$q$','Interpreter','LaTeX','FontSize',22)
ylabel('$\ell(q)$','Interpreter','LaTeX','FontSize',22)
set(gca,'FontSize',18,'FontName','Times')
hold off
print -dpdf resampled_mc.pdf
```

Figure 3: The Matlab code `resampled_mc.m`.

(see Vanneste, 2010)

Relationship with Cramér function:

$$x_n = A_n x_{n-1}, \qquad A_n \in \mathbb{R}^{d\times d} \quad \text{i.i.d.}$$

$$\|x_0\| = 1. \qquad n = 1, \dots, N$$

$$\ell(q) = \lim_{N\to\infty} \frac{1}{N} \log \langle \|x_N\|^q \rangle$$

$$= \lim_{N\to\infty} \frac{1}{N} \log \langle \|A_1 \cdots A_N\|^q \rangle$$

$$\ell(0) = 0.$$

Now consider $h_N = \frac{1}{N} \log \|A_1 \cdots A_N\|$

(finite-time Lyapunov exps.)

$$\rho_N(h) \asymp e^{-Ng(h)}$$

asymptotic equivalence of logs as $N \to \infty$

$$g(h) = \text{Cramér function}$$

$$\bar{h} = \lim_{N\to\infty} H_N = \text{Lyapunov exponent}$$

$g(h)$ is convex with a minimum at $h = \bar{h}$.

Now consider:

$$\langle \|x_N\|^q \rangle \asymp \int e^{Nqh} e^{-Ng(h)} dh \asymp e^{N\ell(q)}$$

For large N, can use Laplace's method:

$$e^{N(qh - g(h))} \sim e^{N \sup_h (qh - g(h))}$$

$$\boxed{\ell(q) = \sup_h (qh - g(h))}$$

So the generalized Lyapunov exponents are the Legendre transform of the Cramér function.

Now consider a function: $f : \mathbb{R}^d \to \mathbb{R}$. Let

$$u_n(x) = \langle f(A_n \cdots A_1 x) \rangle$$

Then:
$$\begin{aligned} u_{n+1}(x) &= \langle f(A_{n+1} A_n \cdots A_1 x) \rangle \\ &= \langle f(A_n \cdots A_1 A x) \rangle \\ &= \langle u_n(Ax) \rangle \quad , \quad u_0(x) = f(x) \end{aligned}$$

With $f(x) = \|x\|^q$, $u_n(x_0) = \langle \|x_n\|^q \rangle$,

$$u_n(x) = \lambda^n \|x\|^q v(\hat{e}), \quad \hat{e} = x/\|x\|$$

Insert: $\langle u_n(Ax) \rangle = \lambda^n \langle \|Ax\|^q v(\underbrace{Ax/\|Ax\|}_{A\hat{e}/\|A\hat{e}\|}) \rangle$

$$= u_{n+1}(x)$$

To get eigenfunction, require:

$$= \lambda^{n+1} \|x\|^q v(\hat{e})$$

$$\langle \underbrace{\frac{\|Ax\|^q}{\|x\|^q}}_{\langle \|A\hat{e}\|^q \rangle} v(A\hat{e}/\|A\hat{e}\|) \rangle = \lambda v(\hat{e})$$

So let: eigenvalue problem on S^{d-1}

$$(\mathcal{L}_q v)(\hat{e}) = \langle \|A\hat{e}\|^q v(A\hat{e}/\|A\hat{e}\|) \rangle$$

We then have the eigenvalue problem

$$\boxed{(\mathcal{L}_q v)(\hat{e}) = \lambda v(\hat{e})} \quad \hat{e} \in S^{d-1}$$

Since $u_n(x_0) = \langle \|x_n\|^q \rangle$,

$$\boxed{\ell(q) = \log \lambda_1}$$

Largest eigenvalue of $\mathcal{L}_q$.

Now let:

$$(\tilde{\mathcal{L}}_q v)(\hat{e}) = \langle \|B\hat{e}\|^q \, v\left(B\hat{e}/\|B\hat{e}\|\right) \rangle$$

with $B = A^{-1}/|\det A|^{1/q}$.

Let's show that $\tilde{\mathcal{L}}$ is related to the adjoint of $\mathcal{L}$ in $L^2(S^{d-1})$.

$$\int_{S^{d-1}} w(\hat{e}) \, \mathcal{L} v(\hat{e}) \, d\hat{e}$$

$$= \left\langle \int_{S^{d-1}} \|A\hat{e}\|^q \, w(\hat{e}) \, v\left(A\hat{e}/\|A\hat{e}\|\right) d\hat{e} \right\rangle$$

Let $\hat{e}' = \dfrac{A\hat{e}}{\|A\hat{e}\|}$, $\quad \dfrac{\partial \hat{e}}{\partial \hat{e}'} = \|A\hat{e}\| A^{-1}$

$$\det \frac{\partial \hat{e}}{\partial \hat{e}'} = \frac{\|A\hat{e}\|^d}{\det A}$$

Hence, $d\hat{e} = \left| \det \frac{\partial \hat{e}}{\partial \hat{e}'} \right| d\hat{e}'$

$$= \frac{\|A\hat{e}\|^d}{|\det A|} d\hat{e}' = \frac{\|A^{-1}\hat{e}'\|^{-d}}{|\det A|} d\hat{e}'$$

In the last step: used $\|A\hat{e}\| = \|A^{-1}\hat{e}'\|^{-1}$, since

$$\|A^{-1}\hat{e}'\|^{-1} = \left\| A^{-1} \frac{(A\hat{e})}{\|A\hat{e}\|} \right\|^{-1} = \|A\hat{e}\|$$

$$\int_{S^{d-1}} w(\hat{e}) \mathcal{L} v(\hat{e}) \, d\hat{e}$$

$$= \left\langle \int_{S^{d-1}} \|A^{-1}\hat{e}'\|^{-q} w\left(A^{-1}\hat{e}' / \|A^{-1}\hat{e}'\|\right) v(\hat{e}') \times \frac{\|A^{-1}\hat{e}'\|^{-d}}{|\det A|} d\hat{e}' \right\rangle$$

$$= \left\langle \int_{S^{d-1}} \|A^{-1}\hat{e}'\|^{\boxed{-q-d}} w\left(A^{-1}\hat{e}' / \|A^{-1}\hat{e}'\|\right) v(\hat{e}') \frac{d\hat{e}'}{|\det A|} \right\rangle$$

Hence, $\mathcal{L}_q^{\dagger} = \tilde{\mathcal{L}}_{-q-d}$

Hence, $\mathcal{L}_q$ and $\tilde{\mathcal{L}}_{-q-d}$ have the same spectrum.

Conclude:

$$\boxed{\ell(q) = \ell^{-}(-q-d)}$$

time-reversed system, $A \to A^{-1}$ ($\det A = 1$)

It follows immediately that $\ell(-d) = \ell^{-}(0) = 0.$

Vanneste also shows that if the metrics are symplectic then

$$\ell(q) = \ell^{-}(-q-d)$$

Symplectic matrices satisfy

$\frac{d}{2} \times \frac{d}{2}$ identity matrix

$$A^T J A = J, \qquad J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \quad (d \text{ even})$$

$$J^{-1} = J^T = -J$$

Note then $A^{-1} = J^{-1} A^T J = -JA^T J$,

since $(-JA^T J) A = -J(A^T JA) = -J^2 = I$.

Hence, $A^{-T} = -JAJ$, so $AJ = JA^{-T}$.

$A^{-T} = (A^{-1})^T$

Let $(\mathcal{J} v)(\hat{e}) = v(J\hat{e})$.

$$(\mathcal{J} \mathcal{L}_q v)(\hat{e}) = \left\langle \|AJ\hat{e}\|^q v\left(AJ\hat{e}/\|AJ\hat{e}\|\right)\right\rangle$$

$$= \left\langle \|JA^{-T}\hat{e}\|^q v\left(JA^{-T}\hat{e}/\|JA^{-T}\hat{e}\|\right)\right\rangle$$

$$= \left\langle \|A^{-T}\hat{e}\|^q v\left(JA^{-T}\hat{e}/\|A^{-T}\hat{e}\|\right)\right\rangle$$

$$= \left\langle \|A^{-T}\hat{e}\|^q (\mathcal{J} v)\left(A^{-T}\hat{e}/\|A^{-T}\hat{e}\|\right)\right\rangle$$

$$= \left(\mathcal{L}_q^{-T} \mathcal{J} v\right)(\hat{e})$$

where $\mathcal{L}_q^{-T}$ is defined as for $\mathcal{L}_q$, but with $A \to A^{-T}$.

So: $$J \mathcal{L}_q = \mathcal{L}_q^{-T} J$$

which implies that $\mathcal{L}_q$ and $\mathcal{L}_q^{-T}$ have the same spectrum!

Hence, $\ell(q) = \ell^{-T}(q)$

But $\ell^{-T}(q) = \ell^{-}(q)$, since the transposes don't matter in

$$\| A_N^{-1} \cdots A_1^{-1} \| \text{ vs } \| A_N^{-T} \cdots A_1^{-T} \|.$$

Conclude: $\ell(q) = \ell(-q-d)$ for symplectic matrices.

Let's finish by showing that

$$\dot{x} = u \quad , \quad \nabla \cdot u = 0 \qquad (21)$$

leads to $\dot{M} = GM$ with $M = \frac{\delta x}{\delta X}$ symplectic.

$$G = (\nabla u)^T.$$

$$\nabla \cdot u = 0 \Rightarrow u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \psi \\ \partial_y \psi \end{pmatrix} = J \cdot \nabla \psi$$

$$\nabla u = J : \nabla\nabla \psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} \psi_{xy} & \psi_{yy} \\ -\psi_{xx} & -\psi_{xy} \end{pmatrix}$$

$$G = (\nabla u)^T = -\nabla\nabla\psi : J$$

$$\Rightarrow \dot{M} = GM = -\nabla\nabla\psi : J$$

Now we want to check: $M^T J M = J$

Satisfied at $t = 0$ $(M = I)$.

$$\frac{d}{dt}(M^T J M) = \dot{M}^T J M + M^T J \dot{M}$$

$$= M^T G^T J M + M^T J G M$$

$$= M^T (G^T J + J G) M$$

$$= M^T (JG - (JG)^T) M$$

Hence we need JG symmetric for $M^T J M = \text{const.}$,

$$JG = -J \, \nabla\nabla\psi \, J \quad \text{obviously symmetric.}$$

Conclude: M is symplectic.

Note that this works for $d > 2$ (d even)
for
$$u = J \cdot \nabla\psi$$

This defines Hamiltonian systems.