Lecture 11: Renovating flows

Consider a two-dimensional linear divergence-free velocity field given by

$$\boldsymbol{u}(\boldsymbol{x}) = R(\theta)\, A\, R^T(\theta) \cdot \boldsymbol{x} \tag{1}$$

where A is a constant traceless matrix and

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

is a rotation matrix. The velocity gradient matrix is then

$$(\nabla \boldsymbol{u})^T = R(\theta)\, A\, R^T(\theta). \tag{3}$$

An infinitesimal line segment $\delta\boldsymbol{x}$ obeys

$$\delta\dot{\boldsymbol{x}} = \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u}\,. \tag{4}$$

Hence, as long as $\nabla\boldsymbol{u}$ remains constant, the initial line segment $\delta\boldsymbol{x}(0)$ is stretched after a time τ to

$$\delta\boldsymbol{x}(\tau) = \exp(\tau R A R^T) \cdot \delta\boldsymbol{x}(0). \tag{5}$$

For any traceless matrix A with determinant $\det A = -\zeta^2$, we have

$$\exp A = I\, \cosh\zeta + A\,\zeta^{-1}\sinh\zeta. \tag{6}$$

Hence,

$$\delta\boldsymbol{x}(\tau) = \big(I\, \cosh\zeta + \tau R A R^T\,\zeta^{-1}\sinh\zeta\big) \cdot \delta\boldsymbol{x}(0). \tag{7}$$

Now let

$$A = \begin{pmatrix} \gamma & -\omega \\ \omega & -\gamma \end{pmatrix}, \tag{8}$$

where γ is the rate-of-strain of the flow, and ω is half its vorticity ($\nabla \times \boldsymbol{u} = 2\omega\hat{\boldsymbol{z}}$). The corresponding rotated matrix is

$$R A R^T = \begin{pmatrix} \gamma\cos 2\theta & \gamma\sin 2\theta - \omega \\ \gamma\sin 2\theta + \omega & -\gamma\cos 2\theta \end{pmatrix}, \tag{9}$$

and the exponential is

$$\exp(\tau R A R^T) = \begin{pmatrix} \cosh\zeta + (\gamma\tau/\zeta)\cos 2\theta\sinh\zeta & ((\gamma\tau/\zeta)\sin 2\theta - (\omega\tau/\zeta))\sinh\zeta \\ ((\gamma\tau/\zeta)\sin 2\theta + (\omega\tau/\zeta))\sinh\zeta & \cosh\zeta - (\gamma\tau/\zeta)\cos 2\theta\sinh\zeta \end{pmatrix} \tag{10}$$

with

$$\zeta = \sqrt{-\det(\tau R A R^T)} = \tau\sqrt{\gamma^2 - \omega^2}. \tag{11}$$

Note that this expression is valid for $\gamma^2 < \omega^2$, as well as for $\gamma^2 = \omega^2$ by taking the limit. The latter case corresponds to a shear flow, since then $A^2 = (\gamma^2 - \omega^2)I = 0$ with $A \neq 0$.

To simplify expressions, we let

$$\Gamma = \gamma\tau/\zeta, \qquad \Omega = \omega\tau/\zeta, \tag{12}$$

whence (10) becomes

$$\exp(\tau RAR^T) = \begin{pmatrix} \cosh\zeta + \Gamma\cos 2\theta\sinh\zeta & (\Gamma\sin 2\theta - \Omega)\sinh\zeta \\ (\Gamma\sin 2\theta + \Omega)\sinh\zeta & \cosh\zeta - \Gamma\cos 2\theta\sinh\zeta \end{pmatrix}. \tag{13}$$

The matrix RAR^T can represent an arbitrary 2D linear flow: there a 3 free parameters (θ, γ, ω), which is the same as the number of independent components of a traceless 2D matrix. Now we assume that the flow *renovates*: for fixed γ and ω, we choose a uniformly-distributed random angle $\theta \in [0, 2\pi)$. We allow this flow to act for a time τ, and after that period we select a new, independent random angle and start over. The random angle θ allows is to make analytic progress, and to compute explicitly quantities such as Lyapunov exponents.

Equation (7) is linear in $\delta\boldsymbol{x}$, so the initial length of $\delta\boldsymbol{x}$ is irrelevant and doesn't have to be infinitesimal. Moreover, the angle θ is random, so we may choose for $\delta\boldsymbol{x}$ a vector $\boldsymbol{\ell} = (1\ 0)$ that lies along the x axis with unit length. Then after one step it is transformed to the vector

$$\boldsymbol{\ell}' = \exp(\tau RAR^T)\cdot\boldsymbol{\ell} = \big(\cosh\zeta + \Gamma\cos 2\theta\sinh\zeta \ \ (\Gamma\sin 2\theta + \Omega)\sinh\zeta\big) \tag{14}$$

which is just the first column of (13). The length of the transformed vector is

$$\begin{aligned} \|\boldsymbol{\ell}'\|^2 &= (\cosh\zeta + \Gamma\cos 2\theta\sinh\zeta)^2 + (\Gamma\sin 2\theta + \Omega)^2\sinh^2\zeta \\ &= \cosh^2\zeta + \Gamma\cos 2\theta\sinh 2\zeta + (\Gamma^2 + \Omega^2 + 2\Gamma\Omega\sin 2\theta)\sinh^2\zeta\,. \end{aligned}$$

To find the Lyapunov exponent, we need to average $\log\|\boldsymbol{\ell}'\|$ over θ. Write

$$\|\boldsymbol{\ell}'\|^2 = a + b\,\sin 2\theta + c\,\cos 2\theta \tag{15}$$

with

$$a = \cosh^2\zeta + (\Gamma^2 + \Omega^2)\sinh^2\zeta, \qquad b = 2\Gamma\Omega\sinh^2\zeta, \qquad c = \Gamma\sinh 2\zeta. \tag{16}$$

The logarithm of the length is then

$$\begin{aligned} 2\log\|\boldsymbol{\ell}'\| &= \log(a + b\,\sin 2\theta + c\,\cos 2\theta) \\ &= \log a + \log(1 + (b/a)\sin 2\theta + (c/a)\cos 2\theta) \\ &= \log a + \log(1 + \alpha\,\cos(2\theta + \beta)) \end{aligned} \tag{17}$$

where β is some phase, and

$$\alpha^2 = (b^2 + c^2)/a^2 = 1 - (\Gamma^2\cosh 2\zeta - \Omega^2)^{-2}, \qquad 0 \le \alpha < 1. \tag{18}$$

Note that α is zero if and only if γ is zero. Now we average over θ:

$$2\langle\log\|\boldsymbol{\ell}'\|\rangle = \log a + \frac{1}{2\pi}\int_0^{2\pi}\log(1 + \alpha\,\cos(2\theta + \beta))\,\mathrm{d}\theta. \tag{19}$$

The phase β is inconsequential, so we drop it and evaluate the integral:

$$\begin{aligned} 2\langle\log\|\boldsymbol{\ell}'\|\rangle &= \log a + \frac{1}{\pi}\int_0^{\pi}\log(1 + \alpha\,\cos\psi)\,\mathrm{d}\psi \\ &= \log a + \log\left(\tfrac{1}{2}(1 + \sqrt{1 - \alpha^2})\right) \\ &= \log\left(\tfrac{1}{2}a\,(1 + \sqrt{1 - \alpha^2})\right). \end{aligned}$$

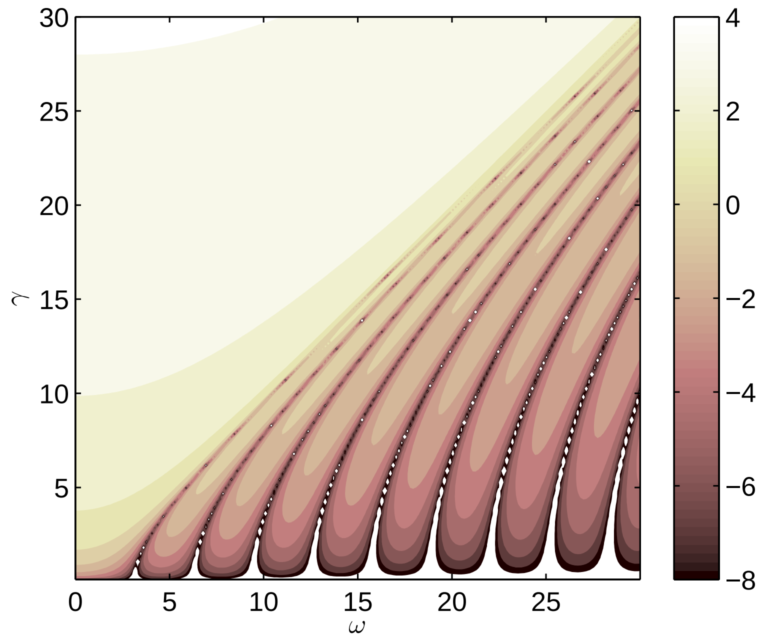

FIG. 1. Contour plot of the logarithm of the Lyapunov exponent (20) for a renovating randomly-oriented linear flow with period $\tau = 1$, as a function of the strain rate γ and half-vorticity ω.

After some manipulation, we obtain the simple form

$$\lambda = \frac{1}{\tau}\langle \log\|\boldsymbol{\ell}'\|\rangle = \frac{1}{2\tau}\log\left(\frac{\gamma^2\cosh^2(\tau\sqrt{\gamma^2-\omega^2})-\omega^2}{\gamma^2-\omega^2}\right), \qquad \gamma > \omega, \tag{20}$$

for the (positive) Lyapunov exponent λ. This is clearly positive for $\gamma^2 > \omega^2$. The expression is also valid for the 'vortical' case $\omega^2 > \gamma^2$, but then it is preferable to write

$$\lambda = \frac{1}{2\tau}\log\left(\frac{\omega^2-\gamma^2\cos^2(\tau\sqrt{\omega^2-\gamma^2})}{\omega^2-\gamma^2}\right), \qquad \gamma < \omega. \tag{21}$$

There are three limiting cases of interest:

(i) For $\omega = 0$, we get the pure-strain limit

$$\lambda = \frac{1}{\tau}\log\cosh(\tau\gamma), \qquad \omega = 0. \tag{22}$$

Since $\cosh|x| < \mathrm{e}^{|x|}$ for $x \neq 0$, we have $\lambda < |\gamma|$ for $\tau\gamma \neq 0$. The reorientation of the axes of stretching due to renovation thus always decreases the stretching that would occur due to constant strain, because it takes some time for our line segment to align itself with the new axes. When $\tau|\gamma| \gg 1$, we recover $\lambda = |\gamma|$, that is, the Lyapunov exponent is equal to the rate-of-strain, since for a long period the segment has plenty of time to re-orient and stretch fully at each period.

(ii) For $\gamma = 0$, we get the pure-rotation limit

$$\lambda = 0, \qquad \gamma = 0, \tag{23}$$

so at least some strain is required to have a nonzero Lyapunov exponent.

(iii) Finally, for $\gamma \to \omega$ we have $A^2 = (\gamma^2 - \omega^2)I = 0$, and we get the shear-flow limit:

$$\lambda = \frac{1}{2\tau} \log\left(1 + \tau^2\omega^2\right), \qquad \gamma = \omega. \tag{24}$$

Note that even though a simple shear flow does not have a positive exponent (its eigenvalues are zero), a renovating shear flow does: it behaves like a hyperbolic system. This highlights the crucial role of re-orientation as a mechanism in chaotic dynamics.

The magnitude of λ as a function of γ and ω is plotted in Fig. 1: Notice the periodic windows where the exponent is zero for $\omega > \gamma$. These occur whenever $\cos^2(\tau\sqrt{\omega^2 - \gamma^2}) = 1$ in (21), or $\tau\sqrt{\omega^2 - \gamma^2} = m\pi$, $m \in \mathbb{Z}$. This corresponds to $\zeta = \mathrm{i}\pi m$ in (10), and leads to $\exp(\tau R A R^T) = (-I)^m$, with obviously no stretching.