Lecture 9: Finite-time Lyapunov exponents

Last time:

$$\lambda_X^{(r)} = \lim_{t\to\infty} \frac{1}{t} \log \| D\varphi_t(X) v \|$$

For ergodic systems, $\lambda_X^{(r)} = \lambda^{(r)}$.

Now consider dropping the limit:

$$\lambda_X^{(r)}(t) = \frac{1}{t} \log \| D\varphi_t(X) v \|$$

Now even for ergodic systems these are a function of X.

Consider the SVD of $D\varphi_t(X)$:

$$D\varphi_t(X) = U_t(X)\, D_t(X)\, V_t^T(X).$$

Drop the t's and X's

$$\| D\varphi_t(X) v \|^2 = v \cdot D\varphi_t^T D\varphi_t \cdot v$$

$$= v \cdot V D^2 V^T \cdot v$$

Now choose $v_i^{(k)} = V_{ik}$, the k^{th} column of V.

$$\| D\varphi_t(X) v^{(k)} \|^2 = v_i^{(k)} V_{im} D_{ml}^2 V_{jl} v_j^{(k)}$$
$$= V_{ik} V_{im} D_{ml}^2 V_{jl} V_{jk}$$
$$= D_{kk}^2 \qquad \text{(no sum over } k\text{)}$$

So even for finite time we can define the eigenspaces. We have also

$$D_{kk} = e^{\lambda_X^{(k)}(t)\, t}$$

It is convenient to derive equations for the time-evolution of the SVD.

Recall: $\frac{d}{dt} D\varphi_t = (\nabla u)^T D\varphi_t$

Let $M = D\varphi_t$, $A = (\nabla u)^T$.

$$\dot{M} = AM \quad , \quad M = UDV^T$$

$$\dot{M} = \dot{U} D V^T + U \dot{D} V^T + U D \dot{V}^T = AM$$

$$(*) \quad U^T \dot{U} D + \dot{D} + D \dot{V}^T V = \underbrace{U^T A U}_{\hat{A}} D$$

Note that $U^T\dot{U}$ is antisymmetric, since

$$\frac{d}{dt}(U^TU) = 0 = \dot{U}^TU + U^T\dot{U}$$

$$(U^T\dot{U})^T = -U^T\dot{U}$$

and similarly for $V^T\dot{V}$.

no sum over j

$$(U^T\dot{U}D)_{ij} = (U^T\dot{U})_{ik}D_{kj} = (U^T\dot{U})_{ij}\,D_{jj}$$

$$(D\dot{V}^TV)_{ij} = D_{ik}(\dot{V}^TV)_{kj} = D_{ii}(\dot{V}^TV)_{ij}$$

Hence, if we evaluate the matrix equation $(*)$ on its diagonal $(i=j)$, we find:

$$\underbrace{(U^T\dot{U})_{ii}}_{0}D_{ii} + \dot{D}_{ii} + D_{ii}\underbrace{(\dot{V}^TV)_{ii}}_{0} = \hat{A}_{ii}D_{ii}$$

(no sum over i)

$$\boxed{\dot{D}_{ii} = \hat{A}_{ii}D_{ii}}$$

Equation for eigenvalues of M (depends on U)

(no sum over i)

Next we need equations for U and V.

Take (*) again but evaluate for $i \neq j$:

$$(1) \quad (U^T \dot{U})_{ij} D_{jj} + \underbrace{\dot{D}_{ij}}_{0} + D_{ii} (\dot{V}^T V)_{ij} = \hat{A}_{ij} D_{jj}$$

($i \neq j$, no sums over i and j)

Interchange i, j:

$$(U^T \dot{U})_{ji} D_{ii} + D_{jj} (\dot{V}^T V)_{ji} = \hat{A}_{ji} D_{ii}$$

$$(2) \quad -(U^T \dot{U})_{ij} D_{ii} - D_{jj} (\dot{V}^T V)_{ij} = \hat{A}_{ji} D_{ii}$$

Now divide (1) by D_{jj} and (2) by D_{ii} and add:

$$\left(\frac{D_{ii}}{D_{jj}} - \frac{D_{jj}}{D_{ii}} \right) (\dot{V}^T V)_{ij} = \hat{A}_{ij} + \hat{A}_{ji}$$

Let $\Delta_{ij} = D_{ii} / D_{jj}$.

$$\boxed{(\dot{V}^T V)_{ij} = \frac{\hat{A}_{ij} + \hat{A}_{ji}}{\Delta_{ij} - \Delta_{ji}} \quad (i \neq j)}$$

The RHS is antisym in $i \Leftrightarrow j$, as required

Note that $(\dot{V}^T V)_{ij} = 0$ for $i = j$.

Can turn this into an equation for $\dot{V}$:

$$\dot{V}_{ij} = -V_{il}(\dot{V}^T V)_{lj}$$

Since:

$$(\dot{V}^T V) V^T = \dot{V}^T$$

$$V(\dot{V}^T V)^T = \dot{V}$$

$$-V(\dot{V}^T V) = \dot{V}$$

Now $\Delta_{ij} = D_{ii} / D_{jj}$

with $D_{ii} = e^{\lambda^{(i)} t}$

Now assume we order the exponents such that

$$\lambda^{(1)} > \lambda^{(2)} > \cdots > \lambda^{(s)}$$

(Reverse from theorem, but more convenient at this point)

Then $\Delta_{ij} = e^{(\lambda^{(i)} - \lambda^{(j)}) t}$

For large t, $\Delta_{ij} \to \begin{cases} \infty, & i < j \\ -\infty, & i > j \end{cases}$

So $\Delta_{ij} + \Delta_{ji} \to \exp(|\lambda^{(i)} - \lambda^{(j)}| t)$

$$+ O(-|\lambda^{(i)} - \lambda^{(j)}| t),$$

$$t \to \infty$$

Hence, $$(\dot{V}V)_{ij} = e^{-|\lambda^{(i)}-\lambda^{(j)}|t}\left(\hat{A}_{ij} + \hat{A}_{ji}\right) + O\left(e^{-2|\lambda^{(i)}-\lambda^{(j)}|t}\right),$$

$t \to \infty$

Conclude:

$\dot{V} \to 0$ exponentially, at a rate given by the gap between exponents.

This will happen even if the $\lambda^{(i)}$ themselves haven't quite converged yet.

This is typical: the Lyapunov exponents often converge slowly, but the characteristic direction on a given trajectory converges very rapidly

What about the U?

Take $(1)/D_{ii} + (2)/D_{jj}$:

$$(U^T\dot U)_{ij}\Delta_{ji} + (\dot V^T V)_{ij} = \hat A_{ij}\Delta_{ji}$$

$$+$$

$$-(U^T\dot U)_{ij}\Delta_{ij} - (\dot V^T V)_{ij} = \hat A_{ji}\Delta_{ij}$$

$$\Rightarrow (U^T\dot U)_{ij}(\Delta_{ji}-\Delta_{ij}) = \Delta_{ij}\hat A_{ji} + \Delta_{ji}\hat A_{ij}$$

$$(U^T\dot U)_{ij} = \frac{\Delta_{ij}\hat A_{ji} + \Delta_{ji}\hat A_{ij}}{(\Delta_{ji}-\Delta_{ij})}$$

($i \neq j$, no sums over i and j)

For large t,

$$(U^T\dot U)_{ij} = \begin{cases} -\hat A_{ji}, & i<j \\ \hat A_{ij}, & i>j \end{cases} \qquad t\to\infty$$

So $\dot U \neq 0 \Rightarrow U$ doesn't converge.