Lecture 8: Lyapunov exponents

$$\dot{x} = u(x(t), t), \qquad x(0) = X$$

$$\Rightarrow \quad x(t) = \varphi_t(X).$$

For $\nabla \cdot u = 0$, φ preserves volume, since

$$d^3 x = \underbrace{\left| \frac{\partial x}{\partial X} \right|}_{1 \text{ if } \nabla \cdot u = 0} d^3 X$$

Hence φ preserves Lebesgue measure.

Theorem (Oseledec Multiplicative Ergodic Theorem)

Let $D\varphi_t$ be the tangent map of a measure-preserving flow φ^t on a compact manifold M.

measure

Then there is a $\Gamma \subseteq M$, $\mu(M) = 1$, such that $\varphi_t \Gamma \subseteq \Gamma$, $t \geq 0$, and for all $X \in \Gamma$:

(1) $\Lambda_X := \lim_{t \to \infty} \left(D\varphi_t^T(X) D\varphi_t(X) \right)^{1/2t}$ exists

(2) Let $\exp \lambda_X^{(1)} < \exp \lambda_X^{(2)} < \dots < \exp \lambda_X^{(s)}$

$s = s(X)$

be the distinct eigenvalues of Λ_X, $\lambda_X^{(r)} \in \mathbb{R}$.

(since $\Lambda_X^T = \Lambda_X$)

(we allow $\lambda_X^{(1)} = -\infty$)

Let $U_X^{(1)}, \dots, U_X^{(s)}$ the corresponding eigenspaces.

$$m_X^{(r)} = \dim U_X^{(r)}.$$

Let $V_X^{(0)} = \{0\}$ and $V_X^{(r)} = U_X^{(1)} \oplus \cdots \oplus U_X^{(r)}$, $r = 1, \dots, s$

Then for $v \in V_X^{(r)} \setminus V_X^{(r-1)}$, $1 \leq r \leq s$:

$$\lim_{t \to \infty} \frac{1}{t} \log \| D\varphi_t(X) v \| = \lambda_X^{(r)}$$

The $\lambda_X^{(r)}$ are called Lyapunov exponents or characteristic exponents.

They are φ_t-invariant: $\lambda_{\varphi_t(X)}^{(r)} = \lambda_X^{(r)}$.

The subspaces $\{V_X^{(r)}\}_{r=0}^{s}$ are nested:

$$\{0\} = V_X^{(0)} \subseteq \cdots \subseteq V_X^{(s)} = \mathbb{R}^d$$

dimension

filtration

and are also φ-invariant: $D\varphi_t(X) V_X^{(r)} = V_{\varphi_t(X)}^{(r)}$

Definition: φ is ergodic if the only measurable sets that are mapped onto themselves ($\varphi_t(E)=E, t>0$) have $\mu(E)=0$ or $\mu(E)=1$.

If φ is ergodic, then $\lambda_x^{(r)}$ and $s(X)$ are constant almost everywhere. (Then drop the X)

So how do we interpret the theorem?

First note that it is a statement of mean exponential growth:

$$\| D\varphi_t v \| \sim e^{\lambda^{(s)} t}, \quad \text{for almost all } v.$$

$t \to \infty$

Why almost all v? Because $V_x^{(s)} = \mathbb{R}^d$.

So essentially all vectors in the tangent space grow exponentially at a rate $\lambda^{(s)}$.

So which vectors grow at $\lambda^{(s-1)}$? The ones that have no projection onto this fastest-growing eigenspace: $V_x^{(s-1)} = \mathbb{R}^d \setminus V_x^{(s)}$.

This is a well-known phenomenon which can be illustrated with a constant matrix:

in space

$$D\varphi_t = \begin{pmatrix} e^{\lambda^{(1)}t} & \\ & e^{\lambda^{(2)}t} \end{pmatrix}, \quad \lambda^{(1)} < \lambda^{(2)}$$

Then $D\varphi_t \begin{pmatrix} u \\ v \end{pmatrix} = u e^{\lambda^{(1)}t} + v e^{\lambda^{(2)}t}$

$\rightarrow u e^{\lambda^{(1)}t}$ as $t \to \infty$,

unless $u = 0$. Then $\to v e^{\lambda^{(2)}t}$ as $t \to \infty$.

The power of Oseledec's thm is that it extends this simple idea to much greater generality.

There is a discrete version of the theorem, which applies to measure-preserving maps:

$$x_n = \varphi_n(X), \quad D\varphi_n(X) \text{ tangent map.}$$

$n \in \mathbb{Z}$

Then $$\lambda_X^{(r)} = \lim_{n\to\infty} \frac{1}{n} \log \| D\varphi_n(X) v \|$$

$$v \in V_X^{(r)} \setminus V_X^{(r-1)}$$

When the map is autonomous, write:

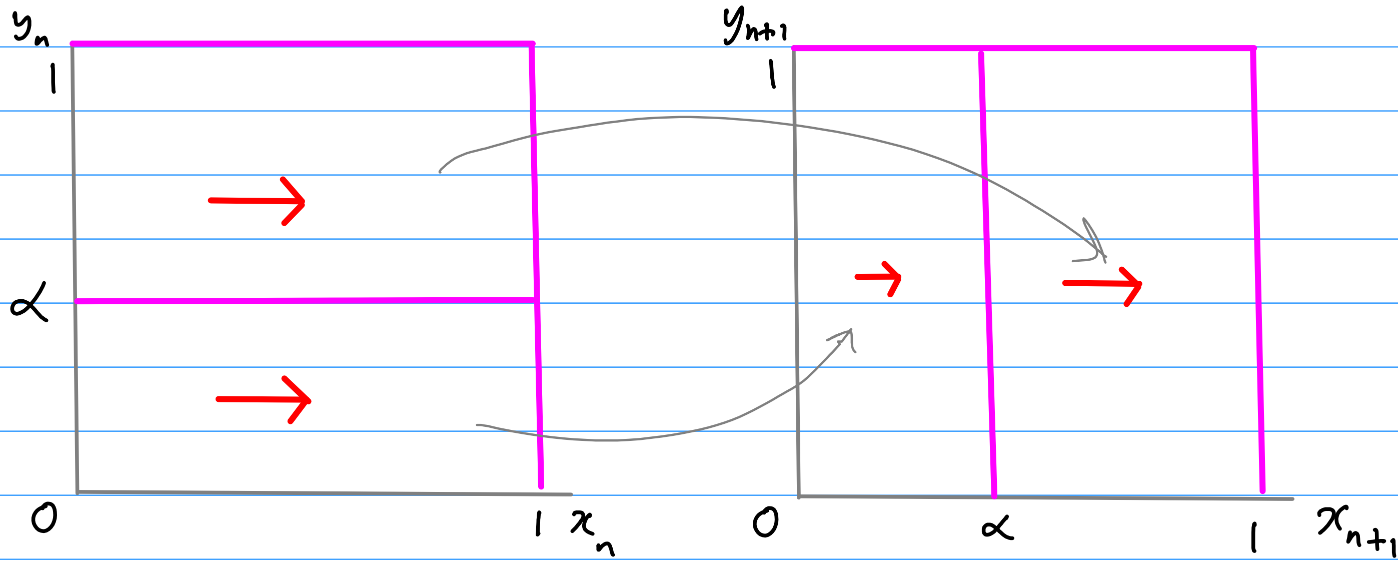
$$\varphi_n(X) = \varphi(\varphi(\varphi \cdots \varphi(X)) = \varphi^{(n)}(X)$$

Example: Baker's map

$X = (x, y) \in [0,1]^2.$ $\quad X_{n+1} = \varphi(X_n)$

$$x_{n+1} = \begin{cases} \alpha x_n & y_n < \alpha \\ \alpha + \beta x_n & y_n > \alpha \end{cases} \qquad \alpha + \beta = 1$$

$$y_{n+1} = \begin{cases} \alpha^{-1} y_n & y_n < \alpha \\ \beta^{-1}(y_n - \alpha) & y_n > \alpha \end{cases}$$



$$D\varphi = \begin{cases} \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}, & y < \alpha \\ \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}, & y > \alpha \end{cases} \qquad \text{area-preserving}$$

Can be shown that this is ergodic

Note that a vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ always shrinks.

All other vectors expand: $V_x^{(2)} = \mathbb{R}^2$, $V_x^{(1)} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$t \in \mathbb{R}$

Now compute Lyapunov exponent:

$$\lambda^{(1)} = \lim_{n\to\infty} \frac{1}{n} \log \| D\varphi_n(X) v \| , \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \| D\varphi(\varphi_{n-1}X) \cdots D\varphi(\varphi_1 X) D\varphi(X) v \|$$

(example: $D\varphi_2(X) = D(\varphi(\varphi(X))) = D\varphi(\varphi(X)) D\varphi(X)$)

Each of the $D\varphi(\varphi_k X)$ is either $\begin{pmatrix} \alpha^{-1} & \\ & \alpha \end{pmatrix}$ or $\begin{pmatrix} \beta^{-1} & \\ & \beta \end{pmatrix}$.

So let $\Lambda_n = r(\varphi_{n-1} X) \cdots r(\varphi_1 X) r(X)$

where $r(X) = \begin{cases} \alpha , & y < \alpha \\ \beta & y > \alpha \end{cases}$

$$\lambda^{(1)} = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \left(r(\varphi_{n-1} X) \cdots r(\varphi_1 X) r(X) \right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \gamma(\varphi_k X)$$

Now we invoke Birkhoff's ergodic theorem:

$$\lambda^{(1)} = \mathbb{E} \log \gamma(X) = \int \log \gamma(X)\, d\mu(X)$$

Since μ is uniform, $\gamma(X) = \alpha$ on a fraction α of the space, β on a fraction β of the space.

$$\boxed{\lambda^{(1)} = \alpha \log \alpha + \beta \log \beta} < 0$$

Since $\det D\varphi = 1$, $\lambda^{(2)} = -\lambda^{(1)} > 0$.

This is a rare case with an analytic expression for λ.