Lecture 7: The tangent map

For typical cases thus far, $\theta \sim e^{-\lambda t}$.

λ comes from the eigenvalues of A in $u = A \cdot x$.

$$A = (\nabla u)^T$$

How do we generalize λ for flows where ∇u is not a spatially-constant matrix?

Go back to trajectories:

$$\dot{x} = u(x(t), t), \qquad x(0) = X$$

Write solutions as $x = \varphi_t(X)$.

The tangent map $D\varphi_t(X) := \dfrac{\partial \varphi_t}{\partial X}(X)$

(This is the same as $\dfrac{\partial x^i}{\partial X^p}$ earlier.)

Recall $\dot{x}$ means $\left.\dfrac{\partial}{\partial t}\right|_X x$, so $\left.\dfrac{\partial}{\partial t}\right|_X$ commutes w. $\dfrac{\partial}{\partial X}$.

holding X constant

Hence, $\dfrac{\partial}{\partial X} \dot{x} = \dfrac{\partial}{\partial t} D\varphi_t = \dfrac{\partial}{\partial X} u(x(t), t)$

$$= \frac{\partial u}{\partial x^i} \underbrace{\frac{\partial x^i}{\partial X^q}}_{(D\varphi_t)^i_q}$$

Hence,

$$\boxed{\frac{\partial}{\partial t} D\varphi_t = (\nabla u)^T \cdot D\varphi_t}$$

This ODE has initial condition $D\varphi_0 = \mathbb{I}$ (identity)

(This equation must be solved together with $\dot{x} = u$.)

For a fixed vector v,

$$\frac{\partial}{\partial t} (D\varphi_t \cdot v) =$$

$$(D\varphi_t \cdot v) \cdot \frac{\partial}{\partial t} (D\varphi_t \cdot v) = (D\varphi_t \cdot v) \cdot (\nabla u)^T \cdot (D\varphi_t \cdot v)$$

$$\frac{1}{2} \frac{\partial}{\partial t} \| D\varphi_t \cdot v \|^2 = (D\varphi_t \cdot v) \cdot e \cdot (D\varphi_t \cdot v)$$

where $e = \frac{1}{2} \left(\nabla u + (\nabla u)^T \right)$

Rate-of-strain tensor

Note that

$$\underbrace{(D\varphi_t \cdot v)}_{V} \cdot e \cdot (D\varphi_t \cdot v) = V \cdot (e \cdot V)$$

$$\leq \|V\| \, \|e \cdot V\| \quad \text{Cauchy-Schwartz}$$

In L^2, we have

$$\|e \cdot V\| \leq \|e\| \, \|V\|$$

when $\|e\|$ is the Frobenius norm of e:

$$\|e\|_F^2 = \sum_{i,j} |e_{ij}|^2.$$

or any matrix norm compatible with vector L^2, such as the induced norm

$$\|e\| = \max_{v \neq 0} \frac{\|e \cdot v\|}{\|v\|} = \text{spectral radius of } e.$$

Hence,

$$\frac{1}{2} \partial_t \|D\varphi_t v\|^2 \leq \|e\| \, \|D\varphi_t v\|^2$$

$$\partial_t \|D\varphi_t v\| \leq \|e\| \, \|D\varphi_t v\|$$

Grönwall's inequality

$$\Rightarrow \|D\varphi_t v\| \leq e^{\int_0^t \|e\| d\tau} \|v\|$$

since $\|D\varphi_0 v\| = \|v\|$.

or $$\boxed{\frac{1}{t}\log\left(\frac{\|D\varphi_t v\|}{\|v\|}\right) \le \frac{1}{t}\int_0^t \|e\| dt \le \sup_t \|e\|}$$

over traj. Replace by space/time.

This at least tells us that for smooth-enough flows, things can't go too crazy.

(Rough ∇u can lead to finite-time blowup of $D\varphi_t$!)

<u>Claim:</u> $$\boxed{\partial_t \det D\varphi_t = (\nabla\cdot u)\det D\varphi_t.}$$

Write $M = D\varphi_t$ to lighten notation ($\dot{M} = AM$), M is $d \times d$

$$\det M = \sum_{i_1\cdots i_d}\sum_{j_1\cdots j_d} \frac{1}{d!}\,\varepsilon_{i_1\cdots i_d}\,\varepsilon_{j_1\cdots j_d}\, M_{i_1j_1}\cdots M_{i_dj_d}$$

time deriv

$$\partial_t \det M = \sum\sum \frac{1}{(d-1)!}\,\varepsilon_{\_}\,\varepsilon_{\_}\,\underbrace{\dot{M}_{i_1j_1}}_{AM,}\, M_{i_2j_2}\cdots M_{i_dj_d}$$

with $A = (\nabla u)^T$

$$\partial_t \det M = \sum_{i_1 \cdots i_d} \sum_{j_1 \cdots j_d} \sum_{l} \frac{1}{(d-1)!} \varepsilon_{i_1 \cdots i_d} \varepsilon_{j_1 \cdots j_d} A_{i_1 l} M_{l j_1} M_{i_2 j_2} \cdots M_{i_d j_d}$$

$$= \sum_{i_1, j_1, l} A_{i_1 l} M_{l j_1} \underbrace{\left(\sum_{i_2 \cdots i_d} \sum_{j_2 \cdots j_d} \frac{1}{(d-1)!} \varepsilon_{i_1 \cdots i_d} \varepsilon_{j_1 \cdots j_d} M_{i_2 j_2} \cdots M_{i_d j_d} \right)}_{\substack{\text{cofactor matrix} \\ = \det M \ (M^{-1})_{j_1 i_1}}}$$

$$= \sum_{i_1 l} A_{i_1 l} \delta_{l i_1} \det M = (\operatorname{trace} A) \det M.$$

Thus, we've finally proved that if trace $A = 0$ $(\nabla \cdot u = 0)$ then $\det D\varphi_t =$ constant $= 1$.

This also makes it clear why $\nabla \cdot u$ controls the "compressibility" of the flow.