Lecture 6: Shear flows

$$\partial_t \theta + x \cdot A^T \cdot \partial_x \theta = \kappa \, \Delta \theta \quad , \quad \operatorname{tr} A(t) = 0.$$

$$B(t) = \int_0^t A(\tau)\, d\tau$$

Consider the two-dimensional case.

Eigenvalues of A = λ and $-\lambda$.

What about $\lambda = 0$? Then either $A \equiv 0$ or A is non-normal:

$$A^2 = 0 \quad , \quad A \neq 0$$

Any matrix A with $\operatorname{tr} A = \det A = 0$ will satisfy $A^2 = 0$.

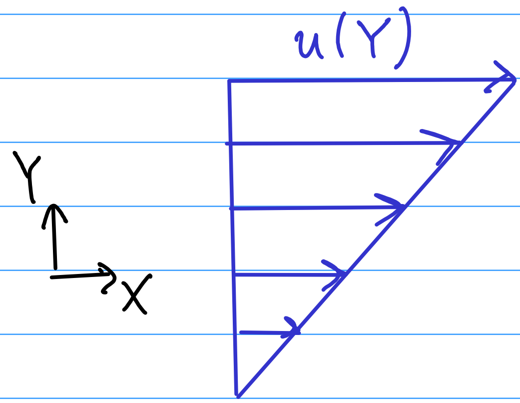
Take, for instance, $A = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$.

$$e^{B} = e^{At} = I + At + \underbrace{\tfrac{1}{2}(At)^2 + \cdots}_{0}$$

$$= I + At$$

So particle trajectories are

$$\underline{x} = e^{B}\underline{X} = \begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X + \alpha t Y \\ Y \end{pmatrix}$$



This is called a <u>shear flow</u>.

These are flows that only vary $\perp$ to their direction.

What is the metric?

$$g = \left(e^{B}\right)^{T}\left(e^{B}\right) = \begin{pmatrix} 1 & 0 \\ \alpha t & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha t \\ \alpha t & 1+(\alpha t)^2 \end{pmatrix}$$

$$g^{-1} = \left(e^{-B}\right)\left(e^{-B}\right)^{T} = \begin{pmatrix} 1 & -\alpha t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\alpha t & 1 \end{pmatrix} = \begin{pmatrix} 1+(\alpha t)^2 & -\alpha t \\ -\alpha t & 1 \end{pmatrix}$$

Hence,

$$\nabla_X \cdot \left(g^{-1} \cdot \nabla_X \Theta\right) = \left(1+(\alpha t)^2\right)\frac{\partial^2 \Theta}{\partial X^2} - 2\alpha t \frac{\partial^2 \Theta}{\partial X \partial Y} + \frac{\partial^2 \Theta}{\partial Y^2}$$

let $\hat{\Theta}(K_x, K_Y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta(X,Y,t)\, e^{-i(K_x X + K_Y Y)}\, dX\, dY$

(Fourier transform)

Then:

$$\partial_t \hat{\Theta} = -\kappa\left(\left(1+(\alpha t)^2\right)K_X^2 - 2\alpha t K_X K_Y + K_Y^2\right)\hat{\Theta}$$

with solution

$$\hat{\Theta}(K_X, K_Y, t) = \hat{\Theta}_0(K_X, K_Y) \times$$

$$\exp\left\{-\kappa\left(\left(t+\tfrac{1}{3}\alpha^2 t^3\right)K_X^2 - \alpha t^2 K_X K_Y + t K_Y^2\right)\right\}$$

$$\Theta(X,Y,t) = \int_{-\infty}^{\infty} \hat{\Theta}_0(K_X, K_Y)\, \frac{e^{i(K_X X + K_Y Y)}}{(2\pi)^2} \times$$

$$\exp\left\{-\kappa\left(\left(t+\tfrac{1}{3}\alpha^2 \underline{t^3}\right)K_X^2 - \alpha t^2 K_X K_Y + t K_Y^2\right)\right\} dK_X\, dK_Y$$

The biggest term in the exponential is t^3.

$$e^{-\frac{1}{3}\kappa\alpha^2 t^3 K_X^2}$$

For large t this kills the integral, unless

$$K_X \sim t^{-3/2}$$

So small wavenumbers get selected in X.

In Y, the dominant term is

$$e^{-\kappa\alpha t^2 K_X K_Y} \sim e^{-\kappa\alpha \underbrace{t^2 t^{-3/2}}_{t^{1/2}} K_Y}$$

So $K_Y \sim t^{-1/2}$.

Given these scalings, can neglect $tK_Y^2 \sim t^{-2}$, but <u>NOT</u> $tK_X^2 \sim 1$.

Now rescale: let $K_X = \xi(\alpha t)^{-3/2}$, $K_Y = \eta(\alpha t)^{-1/2}$:

$$\Theta(X,Y,t) = \int_{-\infty}^{\infty} \hat{\Theta}_0\left(\xi(\alpha t)^{-3/2}, \eta(\alpha t)^{-1/2}\right) \frac{e^{i(\xi(\alpha t)^{-3/2}X + \eta(\alpha t)^{-1/2}Y)}}{(2\pi)^2}$$

$$\times \exp\left\{-\frac{\kappa}{\alpha}\left(\frac{1}{3}\xi^2 - \xi\eta + \eta^2\right)\right\} (\alpha t)^{-2} \, d\xi \, d\eta$$

If we assume $\hat{\Theta}_0(\underset{\sim}{K})\, e^{-\frac{\kappa}{\alpha}(\,)}$ decays for large $|\underline{K}|$ (true even if "rough"), can approximate

$$\hat{\Theta}_0\left(\xi(\alpha t)^{-1/2}, \eta(\alpha t)^{-3/2}\right) \to \underbrace{\hat{\Theta}_0(0,0)}_{\text{this is the average of the initial condition}}$$

Then we can explicitly do the integrals:

$$\Theta(X,Y,t) = \frac{2\pi\sqrt{3}}{(2\pi)^2}\,\frac{(\alpha t)^{-2}}{\chi^2}\,\hat{\theta}_0(0,0)$$

$$\times \exp\left\{-\frac{1}{\chi^2 t^3}\left(3X^2 + 3\alpha t XY + (\alpha t)^2 Y^2\right)\right\}$$

where $\chi = \sqrt{\frac{\kappa}{\alpha}}$ (lengthscale)

Now note that $X = x - \alpha t$, $Y = y$, so

$$3X^2 + 3\alpha t XY + (\alpha t)^2 Y^2 = 3x^2 - 3(\alpha t)xy + (\alpha t)^2 y^2$$

Hence we can write the exponential as

$$e^{-\frac{1}{2}\underline{x}\cdot Q\cdot\underline{x}}$$

where $Q = \frac{1}{\chi^2 t^3}\begin{pmatrix} 3 & -\frac{3}{2}(\alpha t) \\ -\frac{3}{2}(\alpha t) & (\alpha t)^2 \end{pmatrix}$

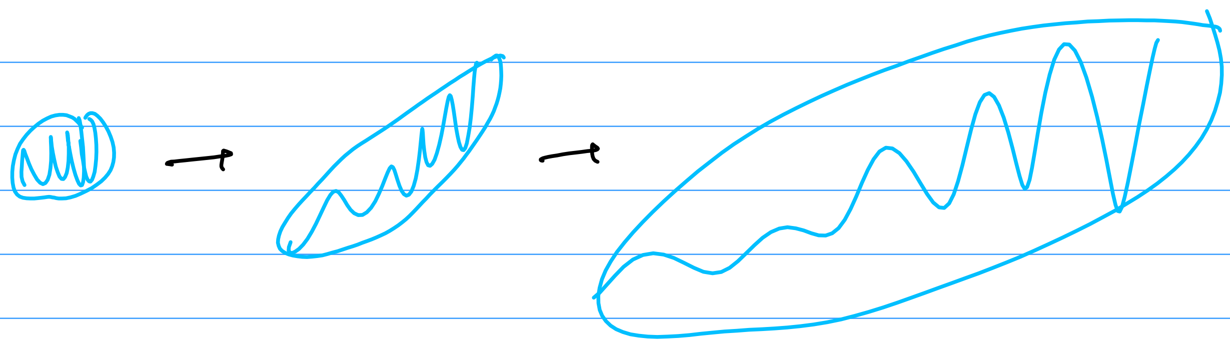
$Q(t)$ has eigenvalues that go as

$$\frac{3}{4\chi^2(\alpha t)^3} \quad \text{and} \quad \frac{1}{\chi^2(\alpha t)} \qquad \text{as } \alpha t \to \infty$$

These correspond to the axes of an ellipsoid:

$$a \sim \frac{2}{\sqrt{3}} \chi (\alpha t)^{3/2}, \quad b \sim \chi (\alpha t)^{1/2}$$

So what happens? A blob tilts in the shear

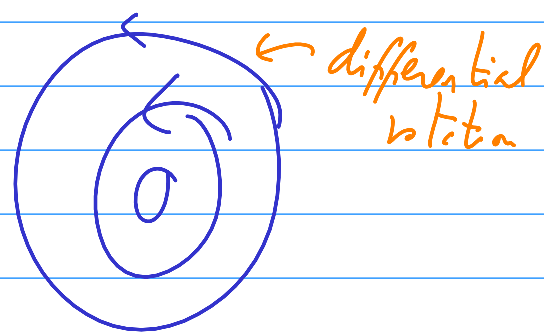


But unlike the exponential case neither direction achieves a constant width. Both axes keep growing, though one at $t^{3/2}$, the other $t^{1/2}$.

This means the area $\sim t^{3/2} t^{1/2} = t^2$,

so by conservation of θ we expect $\theta \sim t^{-2}$, as is indeed the case.

Shear flows are quite common near boundaries and in vortices.



Side note: What is the SVD of B?

$$e^{B} = U D V^{T} \qquad D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \leftarrow \text{eigenvalues of } g$$

$$\Lambda = \frac{1}{2}\left(|\alpha t| + \sqrt{4 + (\alpha t)^2}\right) > 1$$

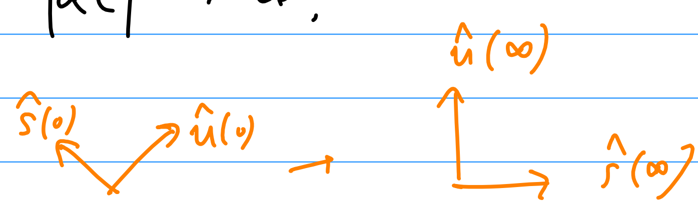
U, V are now time-dependent...

Write as before $\hat{u}, \hat{s}$: $g\hat{u} = \Lambda^{2}\hat{u}$, $g\hat{s} = \Lambda^{-2}\hat{s}$

$$\hat{u} = \frac{1}{\sqrt{1+\Lambda^{-2}}}\begin{pmatrix} \Lambda^{-1} \\ 1 \end{pmatrix}, \qquad \hat{s} = \frac{1}{\sqrt{1+\Lambda^{-2}}}\begin{pmatrix} 1 \\ -\Lambda^{-1} \end{pmatrix}$$

For large t,

$$\left.\begin{aligned} \Lambda &\sim (\alpha t)^2, \\ \hat{u} &\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \hat{s} &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}\right\} \quad |\alpha t| \to \infty.$$



The eigenvectors turn by $\pi/4$ at t goes from 0 to ∞

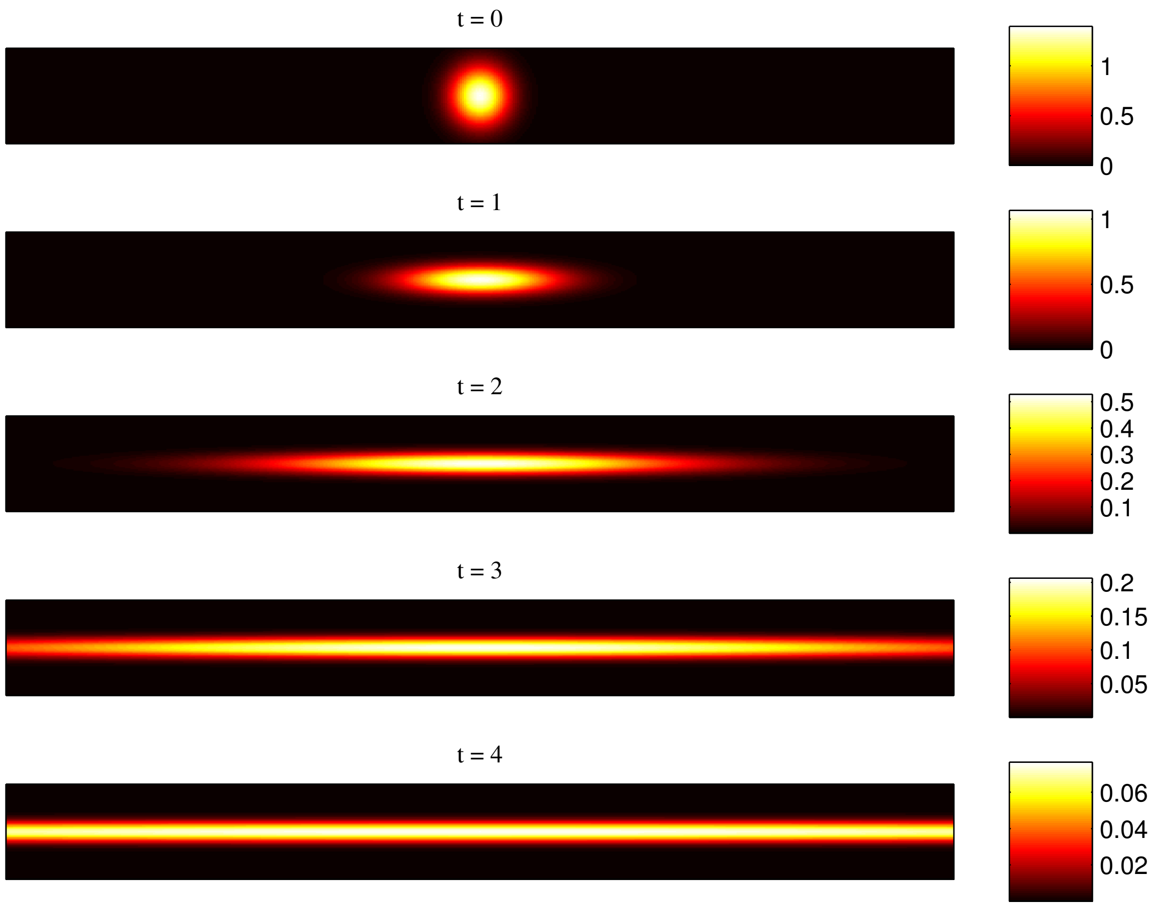


Figure 1: A patch of dye in a uniform straining flow. The amplitude of the concentration field decreases exponentially with time. The length of the filament increases exponentially, whilst its width is stabilised at $\ell = \sqrt{\kappa/\lambda}$. (From J.-L. Thiffeault, *Scalar decay in chaotic mixing*, in Transport and Mixing in Geophysical Flows, J. B. Weiss and A. Provenzale, eds., vol. 744 of Lecture Notes in Physics, Berlin, 2008, Springer, pp. 3–35.)

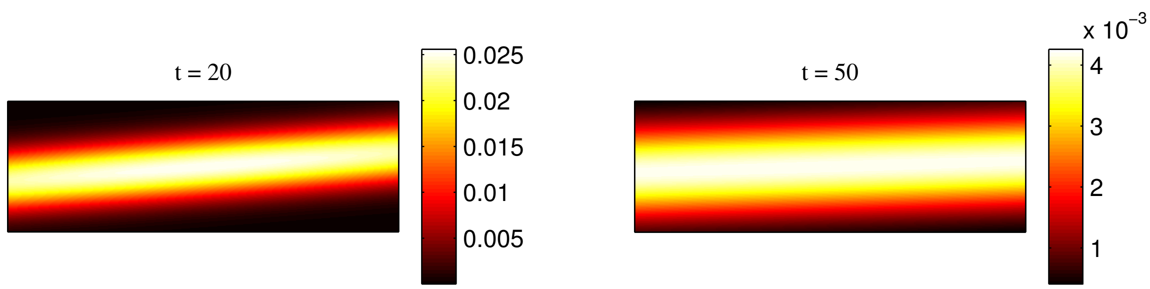


Figure 2: A patch of dye in a uniform shearing flow. The amplitude of the concentration field decreases algebraically with time as t^{-2}. The length of the filament increases as $t^{3/2}$, whilst its width increases as $t^{1/2}$. (From J.-L. Thiffeault, *Scalar decay in chaotic mixing*, in Transport and Mixing in Geophysical Flows, J. B. Weiss and A. Provenzale, eds., vol. 744 of Lecture Notes in Physics, Berlin, 2008, Springer, pp. 3–35.)