Lecture 5: Solving the Lagrangian equation (part 2)

Last time: $e^B = UDV^T$ or $[e^B]^i{}_q = U^i{}_\sigma D_{\sigma\sigma} V_{q\sigma}$

$$\frac{\partial \Theta}{\partial t} = \kappa e^{2|\lambda_s|t} \frac{\partial^2 \Theta}{\partial Y^2} + O(e^{-2\lambda_m t}), \quad t \to \infty$$

$\lambda_s < 0$

Initial condition: $\Theta(\underline{X}, 0) = \Theta_0(\underline{X})$

$$\underline{x} = e^B \cdot \underline{X} = UDV^T \cdot \underline{X}$$

$$(U^T \cdot \underline{x}) = D(V^T \cdot \underline{X}) \quad \leftarrow \text{two frames: } U^T \text{ \& } V^T$$

So let $\underline{x} = U \cdot \underline{\tilde{x}}$, $\underline{X} = V \cdot \underline{\tilde{X}}$

The $\sim$ coordinates are now aligned with the singular eigenvectors:

$$\underline{\tilde{x}} = D\underline{\tilde{X}}$$

(drop tildes)

Redefine (x, y, z) and (X, Y, Z) to be aligned with $\underline{u}, \underline{s}, \underline{m}$, the unstable $(\lambda > 0)$, stable, and middle directions

(Note: since $\lambda_u + \lambda_m + \lambda_s = 0$, $\lambda_u \geq \lambda_m \geq \lambda_s$, we know $\lambda_u > 0$ and $\lambda_s < 0$.)

The coordinates agree at $t=0$, and we have

$$\theta_0(\underline{x}) = \Theta_0(\underline{X}).$$

with $x = X e^{\lambda_u t}$, $y = Y e^{-|\lambda_s| t}$, $z = Z e^{\lambda_m t}$

To solve the advection equation above, define a new time:

$$\kappa e^{2|\lambda_s| t} dt = dT$$

$$T = \frac{\kappa}{2|\lambda_s|}\left(e^{2|\lambda_s| t} - 1\right)$$

$$\Rightarrow \frac{\partial \Theta}{\partial T} = \frac{\partial^2 \Theta}{\partial Y^2} + O\left(e^{-2(\lambda_m + |\lambda_s|)t}\right)$$

Let's assume $\lambda_m + |\lambda_s| > 0$. Then we can neglect the $O(\cdot)$ term for t large, and solve

$$\partial_T \Theta \simeq \partial_Y^2 \Theta.$$

This of course is a simple heat equation, with Green's function

$$G(Y, T; Y_0, 0) = \frac{1}{\sqrt{4\pi T}} e^{-(Y-Y_0)^2/4T}$$

The full solution is then

$$\Theta(X,Y,Z,T) = \int_{-\infty}^{\infty} \Theta_0(X,Y_0,Z)\,\frac{e^{-(Y-Y_0)^2/4T}}{\sqrt{4\pi T}}\,dY_0$$

$$T = \frac{\kappa}{2|\lambda_s|}\left(e^{2|\lambda_s|T}-1\right) = \frac{1}{2}\ell^2 e^{2|\lambda_s|t} + \text{(neglect)}$$

↳ Batchelor length $\ell^2 = \kappa/|\lambda_s|$

Recall that $\theta(x,y,z,t) = \Theta(X,Y,Z,t)$, so

$$\theta(x,y,z,t) = \frac{1}{\sqrt{2\pi}\,\ell}\, e^{-|\lambda_s|t} \int_{-\infty}^{\infty} \theta_0\left(xe^{-\lambda_u t}, Y_0, ze^{-\lambda_m t}\right) \times \exp\left\{-\frac{(ye^{|\lambda_s|t}-Y_0)^2}{2\ell^2 e^{2|\lambda_s|t}}\right\} dY_0$$

We evaluate the LHS at fixed x,y,z, $t\to\infty$.

Hence, $xe^{-\lambda_u t} \to 0$.

Now assume θ_0 has compact support. Then $\theta_0(\cdot, Y_0, \cdot)$ vanishes for $|Y_0|$ large enough.

In that case, for t large enough, ok to replace

$$\exp\left\{-\frac{(ye^{|\lambda_s|t}-Y_0)^2}{2\ell^2 e^{2|\lambda_s|t}}\right\} \quad \text{by} \quad \exp\left\{-\frac{(ye^{|\lambda_s|t})^2}{2\ell^2 e^{2|\lambda_s|t}}\right\}$$

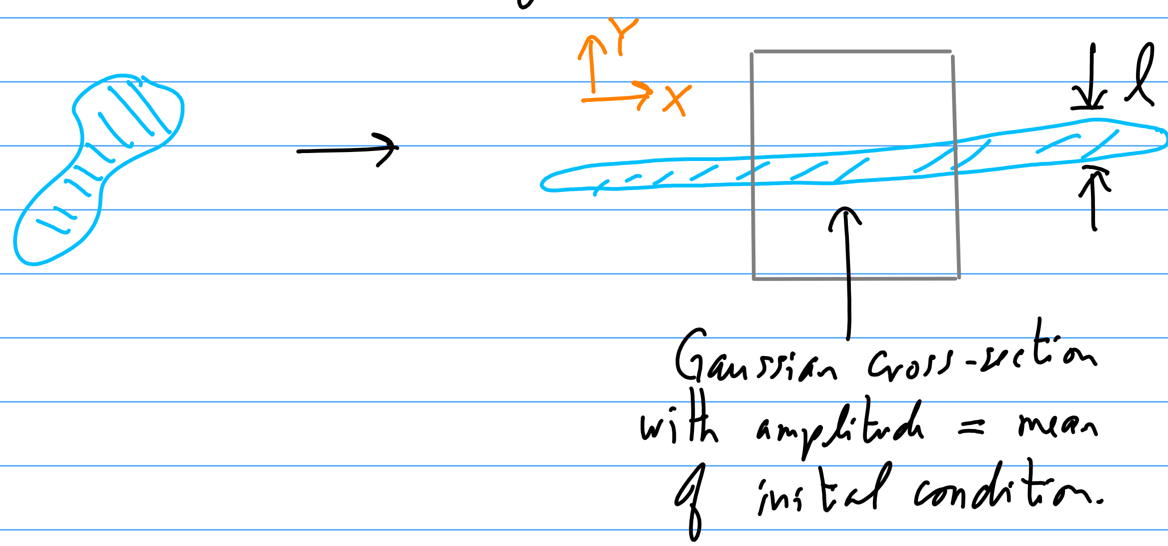
which is equal to $e^{-y^2/2\ell^2}$. Hence,

$$\theta(x,y,z,t) = e^{-|\lambda_s|t}\,\frac{e^{-y^2/2\ell^2}}{\sqrt{2\pi}\,\ell^2}\int_{-\infty}^{\infty}\theta_0(0,Y_0,z\,e^{-\lambda_m t})\,dY_0$$

In 2D, we can drop the 3rd argument of θ_0:

$$\theta(x,y,z,t) = e^{-\lambda t}\,\frac{e^{-y^2/2\ell^2}}{\sqrt{2\pi}\,\ell^2}\int_{-\infty}^{\infty}\theta_0(0,Y_0)\,dY_0$$

Recall our filament solution $e^{-\lambda t}\,e^{-y^2/2\ell^2}$: we have just shown that (assuming our neglect of terms was justified) any compactly-supported initial θ_0 will converge to the filament solution locally.



Note that $\partial_x^2\theta \sim e^{-3\lambda t}$, $\partial_y^2\theta \sim e^{-\lambda t}$,

so $\dfrac{\partial_x^2\theta}{\partial_y^2\theta} \sim e^{-2\lambda t} \to 0$ is always negligible

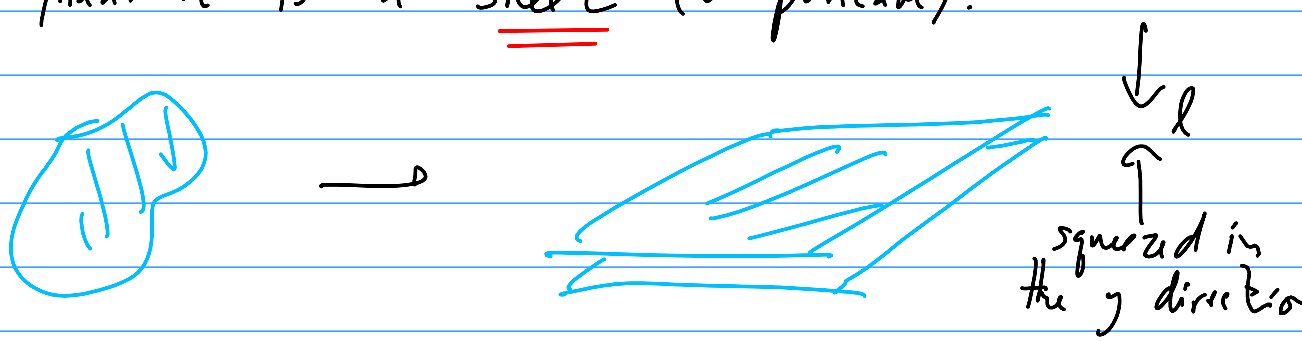
What about 3D? We already assumed $\lambda_m + |\lambda_s| > 0$. But λ_m can have either sign (or 0).

If $\lambda_m > 0$, then the z coordinate is stretched just like the x coordinate, and so $z e^{-\lambda_m t} \to 0$:

$$\theta(x,y,z,t) = e^{-|\lambda_s|t}\,\frac{e^{-y^2/2\ell^2}}{\sqrt{2\pi}\,\ell^2}\int_{-\infty}^{\infty}\theta_0(0,Y_0,0)\,dY_0$$

3D, $\lambda_m > 0$

The filament is a sheet (or pancake):



If $-|\lambda_s| < \lambda_m < 0$, then the z direction behaves like the y direction.

$$\partial_x^2\theta \sim e^{-(|\lambda_s| + 2\lambda_u)t}$$

$$\partial_y^2\theta \sim e^{-|\lambda_s|t}$$

$$\partial_z^2\theta \sim e^{-(|\lambda_s| - 2|\lambda_m|)t}$$

Notice that $\dfrac{\partial_z^2\theta}{\partial_y^2\theta} \sim e^{2|\lambda_m|t} \to \infty$!

So we are not justified in neglecting the $\partial_z^2\theta$ term as $t \to \infty$. Diffusion destroys gradients in the y (fastest contracting) direction, but that leaves gradients in the other contracting direction (z)

We can then show:

$$\theta(x,y,z,t) = e^{-(|\lambda_s| + |\lambda_m|)t} \frac{e^{-\frac{1}{2}\left(\frac{y^2}{\ell_s^2} + \frac{z^2}{\ell_m^2}\right)}}{2\pi \ell_s \ell_m}$$

$$\times \int_{-\infty}^{\infty} \theta_0(0, Y_0, Z_0)\, dY_0\, dZ_0$$

This is a filament with an elliptical cross-section

$$\left(\ell_s = \sqrt{\frac{\kappa}{|\lambda_s|}}\,,\ \ell_m = \sqrt{\frac{\kappa}{|\lambda_m|}}\right)$$