Lecture 4: Solving the Lagrangian equation

Last time: convert the AD eq'n to Lagrangian coordinates:

$$\partial_t \Theta = \kappa \nabla_X \cdot \left(g^{-1} \nabla_X \Theta \right)$$

where $\dot{x} = u(x,t)$, $x(0) = X$

For instance start with linearized velocity equation: ("Filament" lecture)

transpose

$$\partial_t \theta + x \cdot A^T(t) \cdot \nabla_x \theta = \kappa \Delta_x \theta$$

with $u(x) = x \cdot A^T(t)$, $\operatorname{trace} A = 0$.

Trajectories satisfy: $\dot{x} = x \cdot A^T(t)$, $x(0) = X$,

(or $\dot{x} = A(t) \cdot x$)

Try the solution:

$$x(t) = \exp\left(\int_0^t A(\tau)\, d\tau \right) \cdot X$$

The matrix exponential is defined from its Taylor series,

$$e^B = I + B + \frac{1}{2} B^2 + \cdots$$

Check: with $B(t) = \int_0^t A(\tau)\, d\tau$, $\dot{B} = A(t)$,

$$\frac{d}{dt} \exp B(t) = \frac{d}{dt}\left(I + B + \frac{1}{2} B^2 + \cdots \right)$$

$$= \dot{B} + \frac{1}{2}\left(\dot{B} B + B \dot{B} \right) + \cdots$$

If the commutator

$$[B, \dot{B}] = \int_0^t [A(t), A(\tau)]\, d\tau$$

vanishes $\forall t \geq 0$, i.e.,

(1) $$[A(t_1), A(t_2)] = 0, \quad \forall t_1, t_2 \geq 0$$

then we're good, and

$$\frac{d}{dt} \exp B(t) = e^B \cdot \dot{B} = e^B \cdot A$$

$x^i = (e^B)^i{}_j X^j$

so the solution is indeed $x = e^{B(t)} \cdot X$.

Also we then have

$$X = e^{-B(t)} \cdot x$$

With indices: $X^p = [e^{-B}]^p{}_i \, x^i$ ← rows

↖ columns

This immediately gives the deformation gradient,

$$\frac{\partial X}{\partial x} = e^{-B(t)}, \qquad \left(\frac{\partial X^p}{\partial x^i} = [e^{-B}]^p{}_i \right.$$

and the metric $\quad g^{pq} = [e^{-B}]^p{}_i \, [e^{-B}]^q{}_i$

$[e^{-B}]^p{}_i [e^{B}]^i{}_q = \delta^p{}_q$ $\quad$ or $\quad [g^{-1}]^{pq} = \left([e^{-B}][e^{-B}]^T\right)^{pq}$

$$= [e^{-B} e^{-B^T}]^{pq}$$

[It is not a given that

$$e^{-B} e^{-B^T} = e^{-(B+B^T)}$$

This requires $[B(t), B^T(t)] = 0$

The matrix B is then called normal.

$$[B(t), B^T(t)] = \int_0^t [A(\tau), A^T(\sigma)] \, d\tau \, d\sigma,$$

(2) so $[A(t_1), A^T(t_2)] = 0 \quad \forall t_1, t_2 \geq 0$

$\Rightarrow B(t)$ normal.

In general we assume (1) (so we can use the matrix exponential), but not (2).

Progress is made by using the singular value decomposition (SVD) of e^B.

$$e^{B} = UDV^{T}, \qquad U, V \text{ orthogonal}$$
$$e^{-B} = UD^{-1}V^{T}, \qquad D \text{ diagonal} > 0$$

$$[g_{pq}] = e^{B^T} e^{B} = VDU^TUDV^T = VD^2V^T$$

$$[g^{pq}] = VD^{-2}V^{T} \qquad D = \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_n \end{pmatrix}$$

We can also write this in dyad notation:

$$g^{pq} = \sum_{\sigma} V^{p}{}_{\sigma} \Lambda_{\sigma}^{-2} V^{q}{}_{\sigma}$$

$$= \Lambda_u^{-2} u^p u^q + \Lambda_m^{-2} m^p m^q + \Lambda_s^{-2} s^p s^q$$

where $\Lambda_u \geq \Lambda_m \geq \Lambda_s$ (drop Λ_m in 2D)

Note that $\det \sqrt{g} = \Lambda_u \Lambda_m \Lambda_s = 1$

In 2D, $\Lambda_u = \Lambda$, $\Lambda_s = \Lambda^{-1}$.

We call the directions u unstable
m middle
s stable

Insert into the AD equation:

$$\partial_t \Theta = \kappa \nabla_X \cdot \left(g^{-1} \nabla_X \Theta \right)$$

$$= \kappa \nabla_X \cdot \left(\left(\Lambda_u^{-2} uu + \Lambda_m^{-2} mm + \Lambda_s^{-2} ss \right) \cdot \nabla_X \Theta \right)$$

$$= \left[\Lambda_u^{-2} (u \cdot \nabla_X)^2 + \Lambda_m^{-2} (m \cdot \nabla_X)^2 + \Lambda_s^{-2} (s \cdot \nabla_X)^2 \right] \Theta$$

Use the orthogonal directions u, s to define coordinates (X, Y)

$$u \cdot \nabla_X = \frac{\partial}{\partial X}, \quad s \cdot \nabla_X = \frac{\partial}{\partial Y}$$

In the most typical case, Λ_u and Λ_s are exponential in time.

$$\Lambda_s = e^{\lambda_s t} \underset{(2D)}{=} e^{-\lambda t}, \quad \Lambda_u = e^{\lambda_u t} \underset{(2D)}{=} e^{\lambda t}$$

So we can make the approximation

$$\partial_t \Theta = \kappa e^{2|\lambda_s| t} \frac{\partial^2 \Theta}{\partial Y^2} + O(e^{-2\lambda_m t})$$

$t \to \infty$

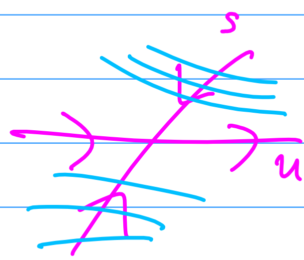
The equation has become one-dimensional

Now we can solve it exactly!

Note that diffusion starts to play a role when

$$\kappa e^{2|\lambda_s| t} = 1 \Rightarrow t = \frac{1}{|\lambda_s|} \log \kappa^{-1}$$

as claimed before.



gradients are amplified in the contracting direction.

Note: the "$t \to \infty$" above should be taken with a grain of salt. It will often be the case that $\partial_Y \Theta \to 0$ rapidly, allowing the other terms to take over.