Lecture 3: Lagrangian coordinates

Goal: show convergence to "filament solution" (last time)

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = \kappa \Delta \theta$$

$\left.\frac{\partial x}{\partial t}\right|_X$

Particle trajectories: $\dot{x} = u(x(t), t)$, $x(0) = X$

Use the X themselves as coordinate.

x — Eulerian or spatial coordinates

X — Lagrangian or material coordinates

$$\theta(x, t) = \Theta(X, t)$$

$$\left.\frac{\partial}{\partial t}\right|_X \Theta(X,t) = \left.\frac{\partial}{\partial t}\right|_X \theta(x,t) = \left.\frac{\partial}{\partial t}\right|_x \theta(x,t) + \nabla\theta \cdot \underbrace{\left.\frac{\partial x}{\partial t}\right|_X}_{u(x,t)}$$

$$= \left.\frac{\partial \theta}{\partial t}\right|_x + u \cdot \nabla \theta$$

Hence, $\left.\frac{\partial \Theta}{\partial t}\right|_X = \kappa \Delta \Theta$

w.r.t. x, not X!

So now transform $\Delta \Theta$ to $\Delta_X \Theta$.

Indices: i, j, k, l Eulerian , p, q, r, s Lagrangian

$$\frac{\partial \Theta}{\partial x^i} = \frac{\partial \Theta}{\partial X^q} \frac{\partial X^q}{\partial x^i}$$

Einstein sum convention (for Benedek)

Define the vectors: $(e^q)_i := \dfrac{\partial X^q}{\partial x^i}$.

$$(e_q)_i := \frac{\partial x^i}{\partial X^q}$$

Note $e^p \cdot e_q = \dfrac{\partial X^p}{\partial x^i} \dfrac{\partial x^i}{\partial X^q} = \delta^p{}_q$

Define also the metric tensor $g_{pq} := e_p \cdot e_q$

Cauchy-Green deformation tensor

right

and its inverse $g^{pq} = e^p \cdot e^q$

left

Check: $g^{pq} g_{qr} = e^p \cdot \underbrace{e^q \, e_q}_{\mathbb{I}} \cdot e_r = e^p \cdot \mathbb{I} \cdot e_q$

$= e^p \cdot e_q = \delta^p{}_q$

Finally, let $g := \det [g_{pq}]$

Claim: $\Delta\Theta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^p}\left(\sqrt{g}\, g^{pq} \frac{\partial\Theta}{\partial x^q}\right)$

$$\Delta\Theta = \frac{\partial}{\partial x^i}\left(\frac{\partial\Theta}{\partial x^i}\right) = \frac{\partial}{\partial x^p}\left(\underbrace{\frac{\partial x^q}{\partial x^i}}_{(e^q)_i} \frac{\partial\Theta}{\partial x^q}\right)\underbrace{\frac{\partial x^p}{\partial x^i}}_{(e^p)_i}$$

$$= \frac{\partial}{\partial x^p}\left(\underbrace{e^p \cdot e^q}_{g^{pq}} \frac{\partial\Theta}{\partial x^q}\right) - \frac{\partial\Theta}{\partial x^q}\boxed{e^q \cdot \frac{\partial e^p}{\partial x^p}}$$

The Christoffel symbols (or connections) are

$$\Gamma^r_{pq} = e^r \cdot \frac{\partial e_p}{\partial x^q} \qquad \left(= e^r \cdot \frac{\partial^2 x}{\partial x^q \partial x^r}, \text{ so } \Gamma^r_{pq} = \Gamma^r_{qp} \right)$$

Since $e^r \cdot e_p = \delta^r{}_p$,

$$\Gamma^r_{pq} = - e_p \cdot \frac{\partial e^r}{\partial x^q}$$

So the boxed term above is

$$e^q \cdot \frac{\partial e^p}{\partial x^p} = e^q \cdot \mathbb{I} \cdot \frac{\partial e^p}{\partial x^p} = \underbrace{e^q \cdot e^s}_{g^{qs}} e_s \cdot \frac{\partial e^p}{\partial x^p}$$

$$= - g^{qs}\,\Gamma^p_{sp}$$

Claim: $\Gamma^{p}_{sp} = \dfrac{\partial}{\partial x^s} \log \sqrt{g}$ metric determinant

First, derive well-known form for Γ:

$$\Gamma^{r}_{pq} = e^{r} \cdot \frac{\partial e_p}{\partial x^q} = e^{r} \cdot e^{s} e_s \cdot \frac{\partial e_p}{\partial x^q}$$

$$= g^{rs} e_s \cdot \frac{\partial e_p}{\partial x^q}$$

since $\dfrac{\partial e_p}{\partial x^q} = \dfrac{\partial e_q}{\partial x^p}$

$$= \frac{1}{2} g^{rs} \left(e_s \cdot \frac{\partial e_p}{\partial x^q} + e_s \cdot \frac{\partial e_q}{\partial x^p} \right)$$

$\dfrac{\partial e_q}{\partial x^s}$ $\quad$ $\dfrac{\partial e_p}{\partial x^s}$

$$= \frac{1}{2} g^{rs} \left(\frac{\partial}{\partial x^q}(e_s \cdot e_p) + \frac{\partial}{\partial x^p}(e_s \cdot e_q) - e_p \cdot \frac{\partial e_s}{\partial x^q} - e_q \cdot \frac{\partial e_s}{\partial x^p} \right)$$

$$= \frac{1}{2} g^{rs} \left(\frac{\partial g_{sp}}{\partial x^q} + \frac{\partial g_{sq}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^s} \right)$$

$$\boxed{\Gamma^{r}_{pq} = \frac{1}{2} g^{rs} \left(\frac{\partial g_{sp}}{\partial x^q} + \frac{\partial g_{sq}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^s} \right)}$$

Now contract r and q:

$$\Gamma^r_{pr} = \frac{1}{2} g^{rs}\left(\cancel{\frac{\partial g_{sp}}{\partial x^r}} + \frac{\partial g_{sr}}{\partial x^p} - \cancel{\frac{\partial g_{pr}}{\partial x^s}}\right)$$

these cancel after relabeling

$$= \frac{1}{2} g^{rs} \frac{\partial g_{rs}}{\partial x^p}$$

Great! Now to show that this equals $\log\sqrt{g}$.

The determinant can be written

$$g = g_{rs} C^{rs} \qquad \text{(no sum over } r\text{)}$$

where C^{rs} is the cofactor matrix of g_{rs}.

$$\frac{\partial g}{\partial g_{rs}} = C^{rs},$$ since the cofactor C^{rs} does not contain g_{rs}.

The inverse of $[g_{rs}]$ is $g^{rs} = \dfrac{C^{rs}}{g} = \dfrac{1}{g}\dfrac{\partial g}{\partial g_{rs}}$

Hence,

$$\Gamma^r_{pr} = \frac{1}{2} g^{rs} \frac{\partial g_{rs}}{\partial x^p} = \frac{1}{2g} \frac{\partial g}{\partial g_{rs}} \frac{\partial g_{rs}}{\partial x^p} = \frac{1}{2g} \frac{\partial g}{\partial x^p}$$

$$\therefore \quad \boxed{\Gamma^{r}_{pr} = \frac{\partial}{\partial x^{p}} \log \sqrt{g}.}$$

So now back to our Laplacian:

$$\Delta \Theta = \frac{\partial}{\partial x^{p}}\left(g^{pq} \frac{\partial \Theta}{\partial x^{q}}\right) + \frac{\partial \Theta}{\partial x^{q}} g^{pq} \frac{\partial \log \sqrt{g}}{\partial x^{p}}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{p}}\left(\sqrt{g}\, g^{pq} \frac{\partial \Theta}{\partial x^{q}}\right)$$

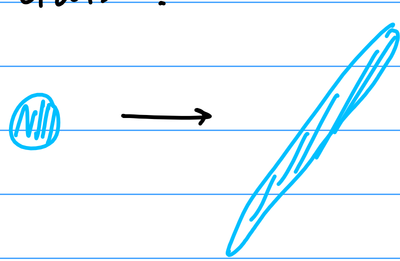
However, $g = 1$ when $\nabla \cdot u = 0$ (show this soon), so we get

$$\boxed{\frac{\partial \Theta}{\partial t} = \kappa \frac{\partial}{\partial x^{p}}\left(g^{pq} \frac{\partial \Theta}{\partial x^{q}}\right)}$$

AD eq'n in Lagrangian coordinates

This holds for any differentiable velocity field!

Seems very simple, but $g^{pq}(X, t)$ can be atrocious!



g_{pq} records how a sphere is locally deformed to an ellipsoid.