Lecture 2: Filament model

Last time: $\dfrac{\partial \theta}{\partial t} + u \cdot \nabla\theta = \kappa \nabla^2 \theta$, $\nabla \cdot u = 0$ (AD)

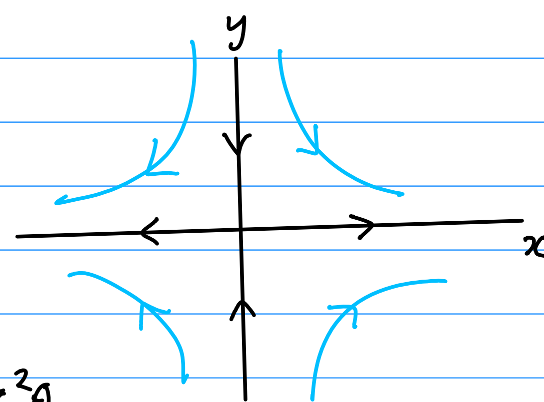
u is stirring (advection)
κ is mixing (diffusion) $\leftarrow$ small at first

Let's look at a simple exact solution that illustrates important features.

Example of a good mixer:

$$\underline{u}(\underline{x}, t) = (\lambda x, -\lambda y)$$

"hyperbolic point"



AD: $\partial_t \theta + \lambda x \partial_x \theta - \lambda y \partial_y \theta = \kappa \nabla^2 \theta$

Can solve this exactly (we'll say more next time), but let's do the simplest thing: look for an x-independent solution of the form:

$$\theta(\underline{x}, t) = e^{-\lambda t} f(y)$$

$$-\lambda f - \lambda y f' = \kappa f''$$

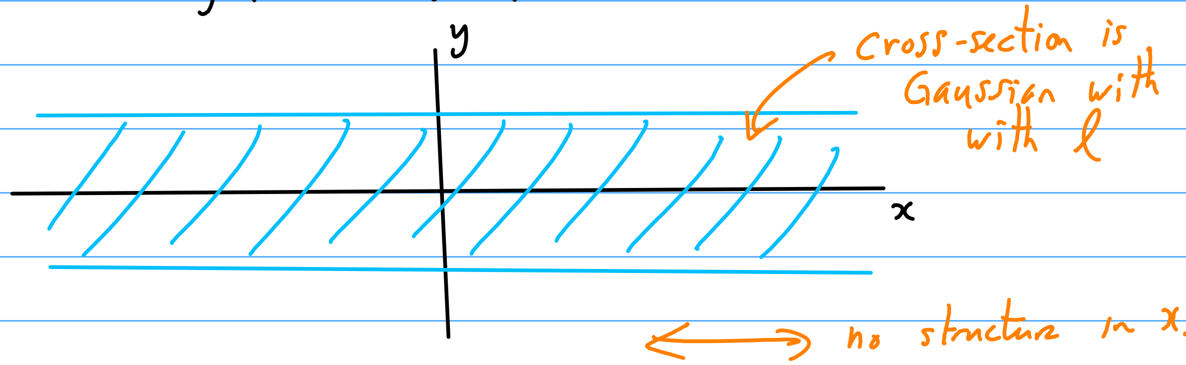
Boundary condition: $f \to 0$ as $y \to \pm\infty$.

Solution is: $f(y) = e^{-y^2/2\ell^2}$, where $\ell^2 = \dfrac{\kappa}{\lambda}$

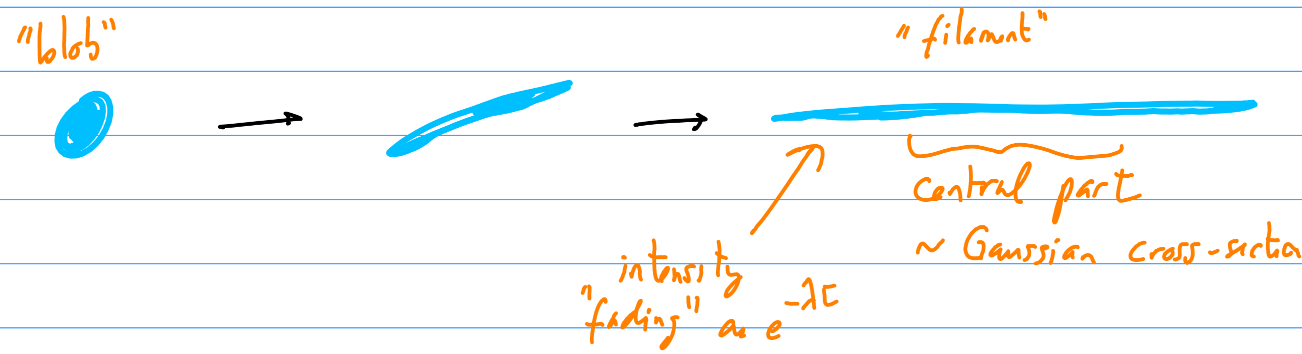
Hence,

$$\boxed{\theta(\underline{x}, t) \sim e^{-\lambda t} e^{-y^2/2\ell^2}}$$

This is the "filament" solution:



In fact, this solution tells us about the ultimate state of any compactly-supported initial condition:



For this case, we know the length scale of "striations":

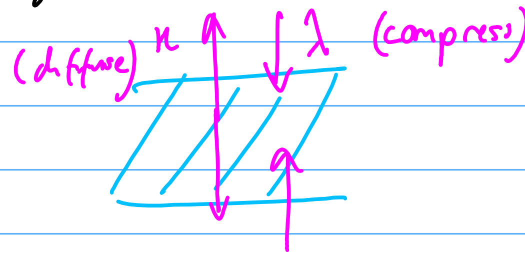
$$\boxed{\ell = \sqrt{\frac{\kappa}{\lambda}}}$$

Batchelor length

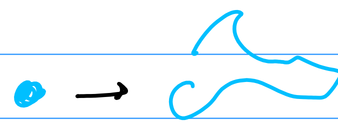
Note $\ell \sim \sqrt{\kappa}$, as necessary to make decay rate indep. of κ!

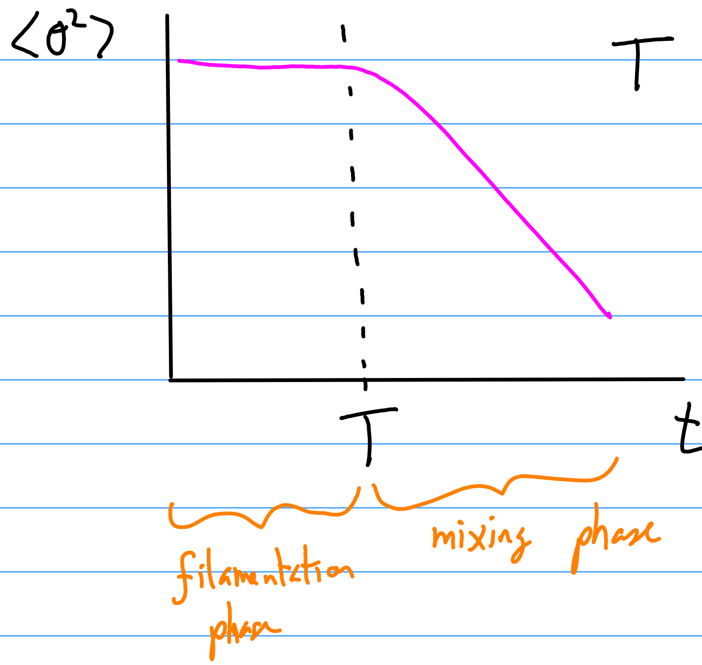
In practical applications, λ is often taken to be the local rate of strain.

ℓ is set by a balance between compression and diffusion



Summary: how mixing proceeds

- A blob is stirred 
- For a while, $\langle \theta^2 \rangle$ is $\sim$ constant, since κ is small
- When $\nabla\theta$ reaches scales of order ℓ, diffusion takes over
- After that, $\langle \theta^2 \rangle$ decays at a κ-independent rate



T given by: $e^{-\lambda T} \sim \sqrt{\kappa}$

$T \sim \lambda^{-1} \log \kappa$

A more general approach:

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla\theta = \kappa \, \Delta\theta$$

Assume smooth $u(x,t)$, choose a reference trajectory:

$$x_0(t), \qquad \dot{x}_0 = u(x_0(t), t)$$

Now let $x = x_0(t) + r$.

Change coordinates from x to r.

$$\theta(x,t) = \tilde{\theta}(r,t)$$

$$\frac{\partial \theta}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{x} \theta(x,t) = \left.\frac{\partial}{\partial t}\right|_{x} \tilde{\theta}(r,t)$$

constant x

$$= \left.\frac{\partial}{\partial t}\right|_{r} \tilde{\theta}(r,t) + \underbrace{\left.\frac{\partial r}{\partial t}\right|_{x}}_{-\dot{x}_0 = -u(x_0,t)} \cdot \nabla_r \tilde{\theta}$$

Also, $u \cdot \nabla\theta = u \cdot \nabla_r \theta$, so

$$\left.\frac{\partial \theta}{\partial t}\right|_{x} + u \cdot \nabla\theta = \left(\left.\frac{\partial \tilde{\theta}}{\partial t}\right|_{r} - u(x_0, t) \cdot \nabla_r \tilde{\theta} \right) + u(x,t) \cdot \nabla_r \tilde{\theta}$$

$$= \frac{\partial \tilde{\theta}}{\partial t} + \underbrace{\left\{ u(x_0 + r, t) - u(x_0, t) \right\}}_{\text{velocity difference between } x \text{ and } x_0} \cdot \nabla_r \tilde{\theta}$$

For smooth u,

$$u(x_0+r,t) - u(x_0,t) = r\cdot\nabla u(x_0,t) + O(r^2)$$

So
$$\frac{\partial\tilde\theta}{\partial t} + r\cdot\nabla u(x_0,t)\cdot\nabla_r\tilde\theta = \kappa\,\Delta_r\tilde\theta + O(r^2)$$

This is an advection-diffusion equation valid in the neighborhood of a fluid trajectory $x_0(t)$.

Drop $\sim$ and o's:

$$\boxed{\frac{\partial\theta}{\partial t} + x\cdot A^T(t)\cdot\nabla\theta = \kappa\,\Delta\theta}$$

AD for linearized velocity field

where $A^T(t) = \nabla u(x_0(t),t)$ is the <u>velocity gradient matrix</u> near $x_0(t)$.

Note that $\nabla\cdot u = 0$ implies

$$\nabla\cdot(x\cdot A^T) = \sum_{i,j}\partial_i(x_j A_{ij}) = \sum_i A_{ii} = \operatorname{tr} A$$

$$\boxed{\operatorname{trace} A = 0}$$

Next we will solve this in general.