Lecture 1: Stirring & Mixing

Stirring: mechanical action   (cause)

Mixing: homogenization of a scalar   (effect)

$\theta(\underline{x}, t)$ = concentration, $\underline{u}(\underline{x}, t)$ given

Advection Diffusion eq.

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad , \quad \nabla \cdot \underline{u} = 0 \quad \text{in} \quad \Omega$$

(AD)

Boundary conditions: $\left. \begin{array}{c} \hat{n} \cdot \nabla \theta = 0 \\ \hat{n} \cdot \underline{u} = 0 \end{array} \right\}$ on boundary $\partial \Omega$

Let $\quad \langle \cdot \rangle = \frac{1}{|\Omega|} \int_{\Omega} \cdot \, dV$ $|\Omega|$ = volume of Ω
          "volume average"

Multiply **AD** by $m \theta^{m-1}$, integrate:

$$\langle m \theta^{m-1} \partial_t \theta \rangle = \partial_t \langle \theta^m \rangle$$

$$\langle m \theta^{m-1} \underline{u} \cdot \nabla \theta \rangle = \langle \underline{u} \cdot \nabla \theta^m \rangle = \langle \nabla \cdot (\underline{u} \theta^m) \rangle \quad \overset{\nabla \cdot \underline{u} = 0}{\nearrow}$$

$$= \int_{\partial \Omega} \theta^m \underbrace{\underline{u} \cdot \hat{n}}_{0!} \, dS = 0$$

$$\langle m \theta^{m-1} \kappa \nabla^2 \theta \rangle = \kappa m \langle \nabla \cdot (\theta^{m-1} \nabla \theta) - \nabla \theta^{m-1} \cdot \nabla \theta \rangle$$

$$= \kappa m \int_{\partial \Omega} \theta^{m-1} \underbrace{\nabla \theta \cdot \hat{n}}_{0!} \, dS - \kappa m (m-1) \langle \theta^{m-2} |\nabla \theta|^2 \rangle$$

$$\partial_t \langle \theta^m \rangle = -\kappa\, m(m-1) \langle \theta^{m-2} |\nabla\theta|^2 \rangle$$

$m = 0$ is trivial

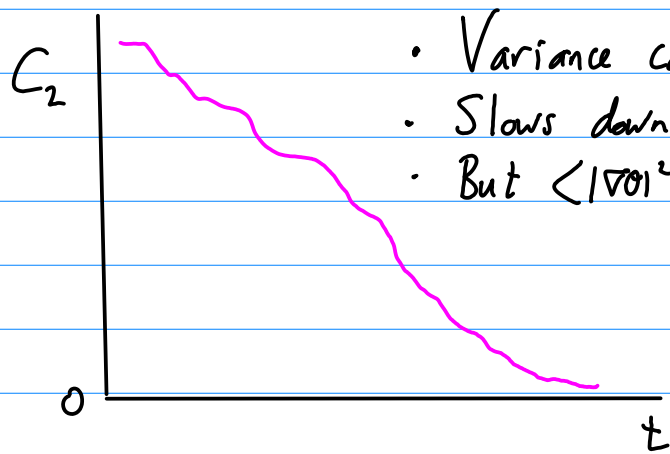$m = 1$: $\quad \partial_t \langle \theta \rangle = 0$ Total amount of θ is conserved

$m = 2$: $\quad \partial_t \langle \theta^2 \rangle = -2\kappa \langle |\nabla\theta|^2 \rangle$ $\langle \theta^2 \rangle$ non-increasing!

Let variance $Var = C_2 = \langle \theta^2 \rangle - \langle \theta \rangle^2$

                                                          ↑
                                                    constant

$$\partial_t C_2 = -2\kappa \langle |\nabla\theta|^2 \rangle$$

Scenario:

- Variance can only decrease.
- Slows down as $\langle |\nabla\theta|^2 \rangle \to 0$
- But $\langle |\nabla\theta|^2 \rangle = 0$ iff $\theta = $ constant.
                                  ↑
                              in some sense

Hence the system is "driven" towards a homogeneous state where.
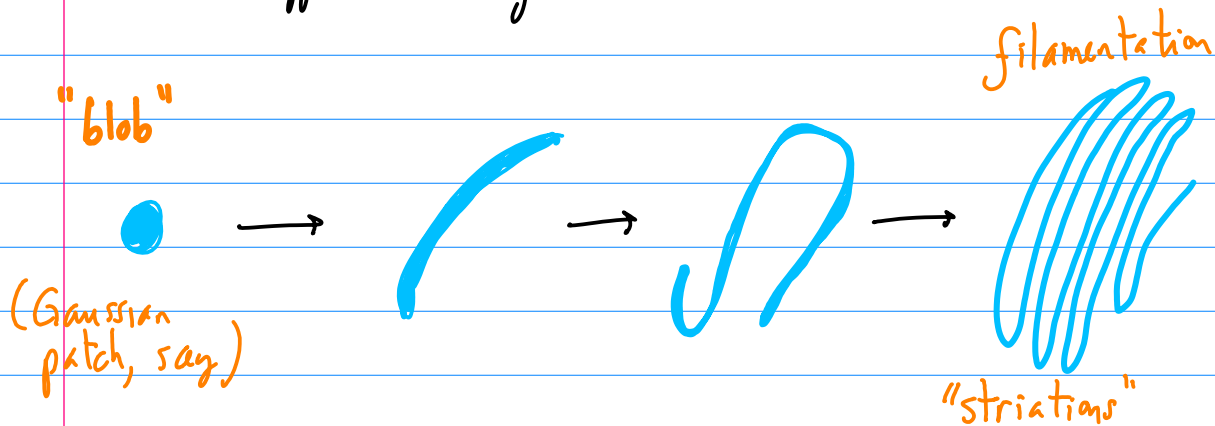
$$\theta(\underline{x}, t) = \langle \theta \rangle = \text{constant}. \qquad \left(C_2 = 0, \ \langle \theta^2 \rangle = \langle \theta \rangle^2 \right)$$

No fluctuations from the mean!   When C_2 is small "enough", we say the system is mixed.

Big Q: Where is $\underline{u}(\underline{x}, t)$!?   (stirring)

    It doesn't appear in the variance equation!

But of course the variance equation is not closed: it depends on $\nabla\theta$.
What happens when you stir?

"blob"

filamentation

(Gaussian patch, say)

"striations"

This hints at the answer: stirring increases $\nabla\theta$

$$\partial_t \langle\theta^2\rangle = -2\kappa \langle |\nabla\theta|^2\rangle$$

this becomes larger as we stir

By how much are gradients increased? After all, if
$|\nabla\theta|$ becomes too large, then $\langle\theta^2\rangle \to 0$, so there are
no gradients anymore!

Answer: for "good" stirring, the system is driven to a
state where
$$\kappa \langle |\nabla\theta|^2\rangle \longrightarrow \boxed{\text{independent of } \kappa}$$

Hence, $\boxed{\nabla\theta \sim \kappa^{-1/2}}$

This is the chaotic/turbulent mixing scenario:

$$\frac{\partial \langle\theta^2\rangle}{\partial t}$$ becomes independent of κ after a "short" transient

This is the Platonic ideal of mixing

(How short? Typically $\sim \log \kappa$)

Furthermore, the smallest scales visible in the concentration field $\theta(x,t)$ have size $\sim \sqrt{\kappa}$. (missing a dimensional factor
→ see later)

Note that $\partial_t \langle \theta^2 \rangle$ independent of κ is crucial: in most applications, κ is tiny!

Heat: $\kappa = 2.2160 \times 10^{-5} \, m^2/s$ at $300K$

10 m room: diffusion time $\sim \dfrac{L^2}{\kappa} = \dfrac{(10m)^2}{(2\times10^{-5} m^2/s)} \sim 4.5 \times 10^6 \, sec$

So we better stir!
Even thermal convection ← ~ 1300 hours
is often enough. $\boxed{\sim 53 \text{ days}!}$