Orbits of Asteroids, a Braid, and the First Link Invariant

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On 22 January 1833, Carl Friedrich Gauss wrote a short passage in one of his mathematical notebooks which was to become widely known among mathematicians and physicists soon after it was first published in 1867:

Of the geometria situs, *which Leibniz foresaw and into which only a pair of geometers (Euler and Vandermonde) were granted the privilege of taking a faint glance, we know and have, after a century and a half, little more than nothing.*

A central problem in the overlapping area of geometria situs *and* geometria magnitudinis *will be to count the intertwinings* [Umschlingungen] *of two closed or infinite curves.*

Let the coordinates of an undetermined point of the first curve be x, y, z; *of the second* x', y', z'; *and let*

$$\iint[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{-3/2}\,[(x'-x)(dydz'-dzdy')+(y'-y)\,dzdx'-dxdz')+(z'-z)(dxdy'-dydx')]=V;$$

then this integral taken along both curves is

$$=4m\pi,$$

m being the number of intertwinings. The value is reciprocal, i.e., it remains the same if the curves are interchanged.[1]

The elusive science of *geometria situs* which Gauss was referring to was soon afterward given the modern name of *Topologie*—topology—by one of Gauss's students, Johann Benedikt Listing.[2] *Geometria magnitudinis*, on the other hand, denoted the kind of analytical geometry which the 18th century had elaborated so impressively, based on the 17th-century ideas of Descartes, Newton, and others. The beautiful formula Gauss wrote down connected the geometry of magnitude with that of position: A linking number, dependent only on the relative positions of two curves in the topological sense, was calculated by an integral involving the coordinates of points on these curves; topological information was extracted from analytical information.

The text of Gauss's fragment poses several historical riddles. As in many other passages of his notebooks, Gauss gave no indication of any proof or argument for his claim, nor did he give any reasons which had led him to consider the linking of space curves at all. Without further information, we cannot even be sure how his claim should be interpreted mathematically: Is it a *definition*; i.e., did Gauss want to say that the possible values of the double integral on the left side of his formula are integer multiples of 4π, and that, therefore, the integer appearing on the right side could be defined as the linking number of the two curves involved? Or is Gauss's formula a *theorem*, computing an independently defined numerical invariant of intertwined curves by analytical means? We thus have the following four questions:

1. When, and how, did Gauss find the integral?
2. How did he know that the values of this integral were integer multiples of 4π?

[1] *Werke*, Vol. V, p. 605. All emphasis in this and the following quotations is in the originals. Square brackets are used to indicate my omissions or additions.

[2] First in a letter of 1836, then in Listing's essay *Vorstudien zur Topologie*, published in 1847. The name *geometria situs*, or *analysis situs*, however, was retained by Riemann and later Poincaré. Only in the first decades of the 20th century, *topology* gradually replaced *analysis situs*. Gauss's reference to Euler is to the latter's *Solutio problematis ad geometriam situs pertinentis* of 1736, dealing with the Königsberg bridges; the reference to Vandermonde is to a paper entitled *Remarques sur les problèmes de situation* of 1771, in which Vandermonde studied various weaving patterns and their symmetries, along with the problem of circuits of knight's moves. Both papers are reprinted in English translation in (Biggs, *et al.* 1976).

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3. Did Gauss think of an independent notion of a linking number?
4. Why did he write down the fragment in early 1833?

The historical literature has not addressed these questions in any detail.[3] Fortunately, it is possible to give substantial answers to all four questions. We will find that the line of thought which eventually induced Gauss to document his insight originated almost 30 years earlier, and in the context of astronomy.

Let us start with the fourth question, and the circumstances of the publication of the passage in 1867, in the fifth volume of Gauss's *Werke.* This volume was edited by Ernst Schering, the Göttingen physicist who was in charge of Gauss's papers until his death in 1897. It was devoted to published and unpublished work on electricity and magnetism. The topological fragment was placed in a section containing other unpublished notes, mainly on electromagnetic induction, and Schering obviously believed that the fragment belonged in this context. His editorial choice was certainly reasonable, though not beyond criticism, as we shall see below.

In fact, in the years following Faraday's discovery of induced currents in 1831, Gauss was working intensely on electromagnetic induction, together with his young friend and colleague, the physicist Wilhelm Weber, who had arrived in Göttingen in September 1831. After more than a year of theoretical and experimental work, the two set up the first telegraph in Germany in Spring 1833, leading from the Göttingen Observatory (in which Gauss both lived and experimented) to Weber's physics laboratory. One of the crucial laws governing the physics of electromagnetic induction was Biot and Savart's law, describing the force which an infinitesimal current element exerted on an "element of positive northern magnetic fluid," as Gauss wrote. His posthumous fragments gave several forms of this interaction. For instance, a current element situated at a point R and directed to an infinitesimally near point R', of strength $\mu = RR'$, acts on a magnetic element at P with a force of strength

$$\frac{\mu \sin R'RP}{(RP)^2}$$

and direction orthogonal to the plane determined by P, R, and R'. Rewritten in modern symbolism, this infinitesimal force is

$$\vec{f} = \frac{d\vec{s} \times \vec{r}}{\|\vec{r}\|^3}$$

if we denote the "vectors" RR' and RP by $d\vec{s}$ and $\vec{r}$, respectively.

This fragment was written in the same notebook as the one on the linking integral and published by Schering immediately following it. Indeed, the integrand of the linking integral is proportional to $\vec{f}d\vec{s}'$ if we choose $d\vec{s}$ as the line element of one of the curves while $d\vec{s}'$ denotes the line element of the other. Thus, there was a compelling electrodynamical interpretation of the linking number: It was proportional to the work V done when a fictitious magnetic test particle was carried along a closed curve in the magnetic field induced by a constant current running through another closed curve. From Ampère's researches, it was known that this work could also be determined by adding the oriented intensities of the currents intersecting a surface bounded by the path of the magnetic test particle.[4] Therefore, it followed that the double integral expressing V was an integer multiple of some constant, independent of the metric details of the situation.

Against this background, it is quite understandable that Schering chose to place the fragment on the linking integral among Gauss's electrodynamical writings. There is even further evidence showing that Gauss was thinking of topological matters in the years of his cooperation with Weber. In 1847, another astronomer and mathematician interested in topology, August Ferdinand Möbius, wrote to Gauss:

> *As I have heard from W. Weber, already some years ago you intended to write a treatise on all possible interlacings* [Verschlingungen] *of a thread, as an introduction to, or preparation for, the theory of electrical and magnetic currents. May we hope that this treatise will soon appear? The fulfillment of this hope would be most desirable for myself and certainly for many others, too.*[5]

Apparently, Gauss was less cautious in his remarks to colleagues than in his own notebooks. No texts have survived which could be regarded as parts of a treatise on topology. Nevertheless, Weber's information was probably right. Well before his electrodynamical work, Gauss had had similar plans to write on *geometria situs* (see below). The resigned tone of the first lines of his fragment on the linking integral may well represent an admission that he was not yet in a position to realize his intentions, according to his high standard of *pauca sed matura.*

All this seemed to confirm the electrodynamic interpretation of Gauss's note. Accordingly, the first readers of Gauss's fragment were physicists. The most important one was in Scotland. Within one year of the publication in 1867, James Clerk Maxwell read Gauss's fragment, and communicated the idea to his scientific friends, including Peter Guthrie Tait in Edinburgh, who was to embark on the classification of knots about 10 years later. Maxwell also reported on it to the London Mathematical Society in early 1869, and worked out the idea of the passage in great detail in his major work, the *Treatise on Electricity and Magnetism* (Maxwell 1873, §§ 409–422). Among other things, he pointed out that there could be nontrivial links

[3]The best treatments of Gauss's topological fragments are still (Stäckel 1918) and (Pont 1974); both give little more than a listing of some relevant sources.

[4]This was an early, magnetostatic particular case of what came to be known as "Stokes's theorem." Gauss discussed this situation in his *Allgemeine Theorie des Erdmagnetismus* of 1838.

[5]Möbius to Gauss, 2 February 1847. The letter is in Gauss's papers and was probably available to Schering. No answer to Möbius's letter seems to be extant. The contact between Weber and Möbius was the consequence of Weber's exile from the Königreich Hannover. With six of his colleagues, Weber had lost his chair at the university in 1837 for his refusal to accept the abolition of Hannover's liberal constitution. In 1843, he accepted a call to Leipzig where Möbius was working.

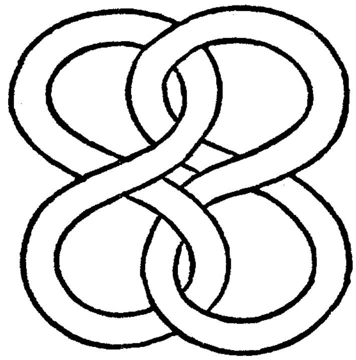

Figure 1. Maxwell's link.

of two components with vanishing linking number, by an example which today is often called the "Whitehead link" (see Fig. 1). Maxwell's discussion closed with a variation on Gauss's theme of the emergence of a new mathematical field:

It was the discovery by Gauss of this very integral, expressing the work done on a magnetic pole while describing a closed curve in presence of a closed electric current, and indicating the geometrical connexion between the two closed curves, that led him to lament the small progress made in the Geometry of Position since the time of Leibnitz, Euler and Vandermonde. We have now, however, some progress to report, chiefly due to Riemann, Helmholtz, and Listing.[6]

Through Schering's edition and Maxwell's reception, the electrodynamical interpretation of the linking integral was well established. If this was the full story, the linking number would have to be *defined* by the double integral, and we should not expect to find references to the situation in question in Gauss's earlier writings. But we do find such references, and they allow us to develop a second way of looking at the mathematics of the linking integral, in a purely geometric fashion.

In order to present the first such reference, I will go back to one of the major breakthroughs in Gauss's scientific career, his calculation of the orbit of the first observed asteroid, Ceres, in 1801. Astronomy held a leading position in the public appreciation of science at the time, and Gauss's success did more to promote his career than his earlier recognition by mathematicians as a leading number-theorist. In the years to follow, astronomers found a large number of similar celestial bodies, and Gauss—still in Braunschweig—continued to think of asteroids. In August 1804, he published a small treatise entitled *Über die Grenzen der geocentrischen Örter der Planeten*, which took up a rather practical question, namely the determination of the celestial region in which a given new asteroid, or planet, might possibly appear.[7] This short article is a striking example of the diversity and density of argument which Gauss was able to achieve in a single text. Published in an astronomical journal, the treatise addressed, at the same time, issues of practical astronomy, such as recent observational data and the making of star maps, *and* mathematical topics in geometry, differential equations, and *geometria situs*.

In fact, Gauss had already been directed to this latter field in the context of his first proof of the fundamental theorem of algebra, in 1799. On 12 October 1802, he addressed the subject in a letter to the astronomer Heinrich Olbers, with whom he had started a correspondence on the occasion of Olbers's observations of Ceres. Gauss mentioned that he expected Carnot's *Géométrie de position* to appear soon, wrongly taking this title in the sense of *geometria situs*. He added:

This still almost unexploited subject, in which we only have a few fragments from Euler and a geometer whom I highly appreciate, Vandermonde, must open a completely new field and form a separate and highly interesting branch of the sublime science of quantity.[8]

Let me take the liberty of presenting the problem of Gauss's paper in modern mathematical language. Let the orbit of the earth's motion around the sun be given by $X \subset \mathbb{R}^3$, and let $X' \subset \mathbb{R}^3$ be the orbit of another celestial body, planet, comet, or asteroid (the sun being at the center of a suitable system of Cartesian coordinates). Determine the region on the sphere given by

$$\left\{ \frac{\vec{x} - \vec{x}'}{\|\vec{x} - \vec{x}'\|} \in S^2 \,\middle|\, \vec{x} \in X, \vec{x}' \in X' \right\}.$$

This region was called *zodiacus* by Gauss. Its determination helped to limit the effort needed both in the observation of the given celestial body and in the production of an atlas of the smallest part of the celestial sphere into which the orbit of the body could be drawn. In order to solve his problem (topologists will already have recognized how it is connected to the linking integral), Gauss derived a differential equation for the *boundary curve or curves* of the *zodiacus*, implicitly assuming the orbits to be smooth curves. If $\vec{x} = (x,y,z)$ and $\vec{x}' = (x',y',z')$ denote the coordinates of orbit points, a necessary condition that a pair of points $(\vec{x},\vec{x}')$ correspond to a boundary point of the *zodiacus* is that the triple consisting of the two tangent vectors to the orbits at $\vec{x}$ and $\vec{x}'$ and the displacement vector $\vec{r} := \vec{x}' - \vec{x}$ be linearly dependent. Gauss expressed this condition by saying that the two tangents at $\vec{x}$ and $\vec{x}'$ had to be coplanar. Translating this condition into a formula led to the differential equation

$$(x' - x)(dy'\,dz - dy\,dz') + (y' - y)(dz'\,dx - dz\,dx') + (z' - z)(dx'\,dy - dx\,dy') = 0.$$

For later use, let us abbreviate the differential form on the left-hand side by ω. Obviously, this form is, up to a change of sign, nothing but the numerator of the integrand in the linking integral! At this point, Gauss inserted a very typical remark: He had under-

[6]From (Maxwell 1873, § 421). That Helmholtz's name appears in this extension of the list of topologists points to another development which had made British physicists aware of topology, namely research on vortex motion in perfect fluids. See my *Topology, Matter, and Space*, to appear in *Archive for History of Exact Sciences*, for a detailed study of this development.

[7]The article is reprinted in *Werke*, Vol. VI, pp. 106–118.

[8]Schilling and Kramer (1900/1909, vol. 1, p. 103).

taken a mathematical study of this equation in its own right, but for the sake of brevity, he did not wish to go into that now.

It turned out that not all solutions of the above differential equation represented actual boundary lines of the *zodiacus*. Gauss distinguished three possible cases: (1) The minimal distance of the celestial body to the sun was greater than the maximal distance between earth and sun, (2) vice versa, (3) the two orbits were *linked*. He showed that in the first two cases, the solutions represented two disjoint closed curves on the sphere, whereas in the third, he found a single closed curve. But, Gauss remarked, none of the regions bounded by this curve could be the *zodiacus* of a case (3) celestial body: In this case, one could show "for reasons of the geometry of position" that the *zodiacus* was the whole celestial sphere. Consequently, the solution of the differential equation $\omega = 0$ now had a different meaning. This, too, Gauss left as a problem to the reader. He remarked that none of the orbits of the known planets was linked with that of the earth, but "comets of the sort exist in abundance."[9] The article closed with a calculation of the *zodiacus* of the recently discovered Pallas and Ceres.

What were the topological reasons alluded to above, and what was the content of Gauss's further study of ω? Gauss kept quiet on this point, but we can make a probable guess on the basis of his later remarks concerning this differential form.[10] Gauss seems to have thought of something like the following geometrical situation: If one looks at the image of a closed orbit X' of a celestial body on the celestial sphere of the earth, with the earth's position fixed at a point $\vec{x}$, one gets an oriented, closed curve that may be denoted by $\gamma_{\vec{x}}$. This curve encloses an *oriented area*, $A(\vec{x})$, which measures what Gauss in later writings called the "solid angle" enclosed by $\gamma_{\vec{x}}$. As with plane angles, this area is well defined only up to a multiple of the total surface of the sphere, 4π. [In modern terms, $\gamma_{\vec{x}}$ is a 1-cycle on the sphere; hence, it bounds a 2-chain $\alpha_{\vec{x}}$, $\gamma_{\vec{x}} = \partial\alpha_{\vec{x}}$. The area $A(\vec{x})$ in question is the integral of the canonical area form over this chain. Since $\alpha_{\vec{x}}$ is well defined only up to a 2-cycle, i.e., up to a multiple of the sphere itself, $A(\vec{x})$ is determined only up to a multiple of 4π.] When the earth moves a small distance along its orbit, say from $\vec{x}$ to $\vec{y}$, the curve $\gamma_{\vec{x}}$ is continuously deformed into a nearby curve $\gamma_{\vec{y}}$, producing a continuous change ΔA of the area enclosed. This area change can be calculated analytically. Up to a sign, it is given by the integral

$$\Delta A = \int_{\vec{x}}^{\vec{y}} \int_{X'} \frac{\omega}{r^3},$$

where r denotes the distance between the two integration arguments of ω, and the first integration follows the earth's orbit between $\vec{x}$ and $\vec{y}$. When the earth has completed one revolution, the planet's orbit appears again at its original position. Therefore, the area change associated with a complete revolution must be an integer multiple of the total surface of the sphere. If this multiple is different from zero, the *zodiacus* of the planet is the whole sphere. In the case of linked ellipses, it is easy to see that the whole sphere is covered when the earth moves around its orbit once. But since this property does not depend on the "measure," i.e., metric structure of the whole setting, it pertains to *geometria situs*.

Here, we have a second, geometric way of deriving the linking integral and its behavior.[11] In modern terms, we may describe it as the calculation of the mapping degree of the mapping defining the *zodiacus*,

$$\Phi : X \times X' \to S^2, \quad (\vec{x}, \vec{x}') \mapsto \frac{\vec{x} - \vec{x}'}{\|\vec{x} - \vec{x}'\|}.$$

Of course, we should be cautious with such reconstructions. But it is evident that Gauss knew a good deal more than he wrote down in his article of 1804. Even if he was not yet thinking in terms of linking numbers, etc., we may conclude that the phenomenon of linking and the relevant differential form were known to Gauss at this time.

Before leaving the article on the *zodiacus*, let me add a remark on the notion of oriented area upon which this reconstruction hinges. In a letter to Olbers of 1825, Gauss mentioned that he had known the notion of an oriented area of plane figures for about 30 years. The crucial problem was to deal with self-intersections of their "circumference," i.e., to determine the area of a figure outlined by an arbitrary closed curve with a finite number of transverse double points. In this case, the different regions of the plane separated by the arcs of the given curve had to be weighted with appropriate coefficients. Gauss explained some examples of this phenomenon—today captured in the homological terminology of chains, cycles, and boundaries—in his letter to Olbers.[12] This letter belongs to the period in which Gauss worked on his *Disquisitiones generales circa superficies curvas* (1827). There, the notion of oriented area was needed for the calculation of the total curvature of a curved surface in 3-space, defined as the oriented area of its image on the sphere under what is usually called the "Gauss map" today. In this way, Gauss was led to reconsider the geometry of closed plane curves (or of closed curves on the

[9] *Werke*, Vol. VI, p. 111f. In 1847, Listing counted 25 pairs of asteroids, whose orbits were known to be linked by then (Listing 1847, p. 64f.).

[10] See in particular *Allgemeine Theorie des Erdmagnetismus* of 1838, in Vol. V of the *Werke*, § 38.

[11] This geometric interpretation was first defended against Schering's electrodynamical interpretation by Schering's student, Otto Böddicker. In his lectures on potential theory in 1874/75, Schering had communicated Gauss's result to his students, among them Böddicker, who decided to choose the linking integral as a dissertation topic. To a large extent, his dissertation (Göttingen, 1876) is a lengthy elaboration of the geometric interpretation of the linking integral, based on the discussion of solid angles in Gauss's *Allgemeine Theorie des Erdmagnetismus*. Note that Maxwell also included the geometrical interpretation in his treatment of the linking integral (Maxwell 1873, §§ 417–421).

[12] Gauss to Olbers, 30 October 1825, in *Werke*, Vol. VIII, p. 398.

sphere) in some detail, and it was in this period that he apparently first thought of preparing a treatise on topological themes. This is again documented in his letters to Olbers, and, most explicitly, in a letter to Schumacher:

Some time ago I started to take up again a part of my general investigations on curved surfaces, which shall become the foundation of my projected work on higher geodesy. [. . .] *Unfortunately, I find that I will have to go* very far *afield since known things must also be developed in a different form, adapted to my investigations. One has to follow the tree down to all its root threads, and some of this costs me week-long intense thought. Much of it even belongs to* geometria situs, *an almost unexploited field. The wish I have always had in all my works, to give them such perfection* ut nihil amplius desiderari possit *[that nothing more can be desired], makes it even more difficult, as well as the necessity to leave my work for other matters.*[13]

The published treatise on curved surfaces did *not* contain the material referred to, but Gauss signaled his intention to pursue these matters further at the end of the sixth paragraph of the *Disquisitiones*, dealing with the calculation of total curvature.

It should be emphasized that *these* considerations, rather than the problem of the classification of knots, formed the background to Gauss's attempts to study the topology of closed plane curves, documented in some fragments on *geometria situs* which were published in Vol. VIII of the *Werke* in 1900.[14]

Late in his career, Gauss returned to the astronomical problem of the limits of the *zodiacus*. In January 1848, at the age of 70, he published a short note in the *Astronomische Nachrichten*.[15] Again, a new asteroid, Iris, had appeared in the firmament, and Gauss's assistant at the Göttingen observatory, Goldschmidt, had calculated its *zodiacus*. Gauss himself did not miss the occasion to communicate some new mathematics. In his note, Gauss tells us in his 44-year-old article on the limits of the *zodiacus* that there had been a problem which he had not dealt with on the earlier occasion, "because a further discussion would only have been a *hors d'oeuvre* there, and because I really wanted to let other people have the pleasure of occupying themselves with a mathematical problem which in my opinion was not uninteresting." The problem he was speaking of was that of the exceptional solutions of the differential equation $\omega = 0$, i.e., those which do *not* represent boundary components of the *zodiacus*. What was their geometrical meaning? Now Gauss gave the answer: A given geocentric position p of a celestial body could arise in different ways; in other words, there might be more than one pair of positions $(\vec{x},\vec{x}')$ which was mapped to $p = \Phi(\vec{x},\vec{x}')$. Implicitly supposing everything to be smooth and generic in an appropriate sense, Gauss pointed out that the celestial sphere was divided into several regions, each of which was covered a different number of times by the mapping Φ. The solutions of the differential equation $\omega = 0$ represented the set of singular points of this branched covering, i.e., the lines where the number of preimages changed by 2. Along the solution curves, an intermediate number of preimages existed. (Of course, Gauss did not speak of "branched coverings" and their singular points, but he described the geometrical situation very clearly.) Gauss ended with a qualitative discussion of the case of two conic sections as orbits; here, the maximum number of preimages of Φ is 4; the different configurations obtained depend on the relative position of these orbits. For instance, in the case of linked ellipses, $\omega = 0$ is satisfied along a single closed curve. Therefore, Gauss concluded his note: in such a case, there existed two regions in the celestial sphere, one of which was covered once, the other thrice.

We can take this second paper on the *zodiacus* as further evidence for the view that already in 1804 Gauss had a rather detailed picture of the geometrical meaning of his differential form, ω.

Up to this point, we have answered questions 4, 1, and 2 relating to the fragment on the linking integral. We have seen two interpretations, one physical and one geometrical, of the linking integral, which explain how Gauss knew that its values were integer multiples of 4π. The crucial third question remains: Did Gauss think of an independent notion of the linking number of two space curves so that his formula represents a computation rather than a definition? A search in Gauss's mathematical notebooks in summer 1994 brought to light an unpublished page on a topological topic which reveals the answer.[16] The editor of Gauss's fragments on *geometria situs*, Paul Stäckel, did not see its importance in 1900, for the simple reason that the mathematical world had not yet directed its attention to the type of mathematical objects discussed by Gauss on this page: braids. Even today, it seems to be a widespread opinion that braids were introduced into mathematics by Emil Artin in 1926, despite Wilhelm Magnus's repeated indications that Adolf Hurwitz had studied the braid group (without its name) in a seminal paper on Riemann surfaces in 1891.[17] A close reading of Tait's papers of 1877–1885 and Listing's *Vorstudien* reveals, however, that braidlike objects had been of interest to both of

[13]Gauss to Schumacher, 21 November 1825, in *Werke*, vol. VIII, p. 400.

[14]Weil (1979) has called attention to an interesting letter of 1863 in which Betti reported that Riemann had told him about some attempts by Gauss to study knots. Riemann's communication, however, refers only to the last years of Gauss's life, i.e., the period *after* Listing had discussed the knot problem in his *Vorstudien zur Topologie*. No documents seem to survive in Gauss's papers which definitely relate to studies of *knots* rather than *closed plane curves*.

[15]Reprinted in *Werke*, Vol. VII, pp. 313–316.

[16]NSUB Göttingen, Cod. Ms. Gauss, Handbuch 7, p. 283.

[17]For a reference to Artin (1926), see (Burde and Zieschang 1985, p. 161). Magnus's allusions to (Hurwitz 1891) can be found in his survey on braid groups (Magnus 1974) and in his book with Chandler (Chandler and Magnus 1982).

them.[18] Given that Listing had been in direct contact with Gauss, we cannot exclude the possibility that these ideas were connected with the even earlier ones of Gauss, which I will now present.

On the page in question is nothing less than a nicely drawn picture of the four-strand braid which we would write as $\sigma_3\sigma_1\sigma_2^{-2}\sigma_3$, using Artin's presentation of the braid group, together with some mathematical comments. It probably belongs roughly to the period of the *Disquisitiones circa superficies curvas*. In the notebook, it immediately precedes some pages with geodetical calculations, dated 1830. On the preceding pages, we find further, undated, geodetical calculations. The last date appearing on previous pages is 1815; apparently, the notebook in question had been out of use for long periods of time. On the back of the page, there is a reading of a weaving pattern of bands. The signs "St" above and below the passage are Stäckel's, showing that he had seen it while editing volume VIII of the *Werke*. Since this fragment has never been discussed in the literature, I will give a detailed interpretation with a complete translation as I go along.

The drawing shows that Gauss thought of the braid as being divided into six segments, extending from one crossing to the next. Gauss numbered these segments and labeled the four strands a, b, c, and d. Then he wrote down a table with the title "change of coordination." Its rows correspond to the labeled strands, its columns to the numbered segments.

a	1	1	$2+i$	$3+i$	$2+2i$	$2+2i$
b	2	2	1	1	1	1
c	3	4	4	4	4	3
d	4	$3+i$	$3+i$	$2+2i$	$3+2i$	$4+3i$

Obviously, Gauss attempted to develop a notation for braids keeping track of (1) the permutations of the strands as one follows the braid and (2) the twists of the strands around each other. The real parts of the numbers appearing in the table specify the positions of the strands, and the imaginary parts were intended to record the twists. The assignment of an i to one of the strands at a crossing of the diagram is ambiguous, however; there seems to be no definite convention adopted in the table (this can be seen already at the first two crossings). The ambiguity is probably due to the fact that Gauss did not yet distinguish between the two possible *orientations* of a half-twist of two strands. That Gauss used complex integers to encode the composition of the braid might be motivated by his well-known fascination with these numbers; it might, however, also have a more serious reason (see below).

Gauss himself seems to have been dissatisfied with his table. Immediately below it, he wrote

What matters is to represent the whole [Inbegriff] *of the entanglement* [Verwicklung] *in such a way as the aggregate of its parts that one sees which parts destroy each other.*

In "parts" of a braid which may "destroy" each other, we can perceive the idea of a composition of braids and of braid equivalence. The meaning of the sentence is thus: Starting from its elementary parts (involving only one crossing), find a representation of a braid as a whole which allows one to decide whether it (or any part of it) is trivial or not. In other words, this passage contains a first formulation of the classification problem for braids (or, more anachronistically, of the word problem in the braid group).

The next remark formulated a conjecture:

Probably it will suffice to list the half twists of one line around the other according to a certain sense of rotation.

Here, Gauss came back to the idea underlying his table, now explicitly addressing orientation. The conjecture can be read in two ways. A *minimal reading* is to understand it as a new proposal for a concise notation of braids; for a stronger reading, see below. In fact, Gauss returned to his example and wrote down

$$cd,\ ab,\ da,\ ad$$

—but then he stopped and struck out the whole line. Also, the second try ended without success; we read

$$cd,\ ab\,.$$

It seems clear what happened. Gauss stumbled at the third crossing, where the orientation of the half-twists of the strands changes for the first time (in modern terms: the inverse of a gen-

[18]See Listing's discussion of "vielfache Helikoiden" and "Helikoiden höherer Ordnungen" (Listing 1847, pp. 43–51), or Tait on "clear coils" (Tait 1877, §§ 25, 26).

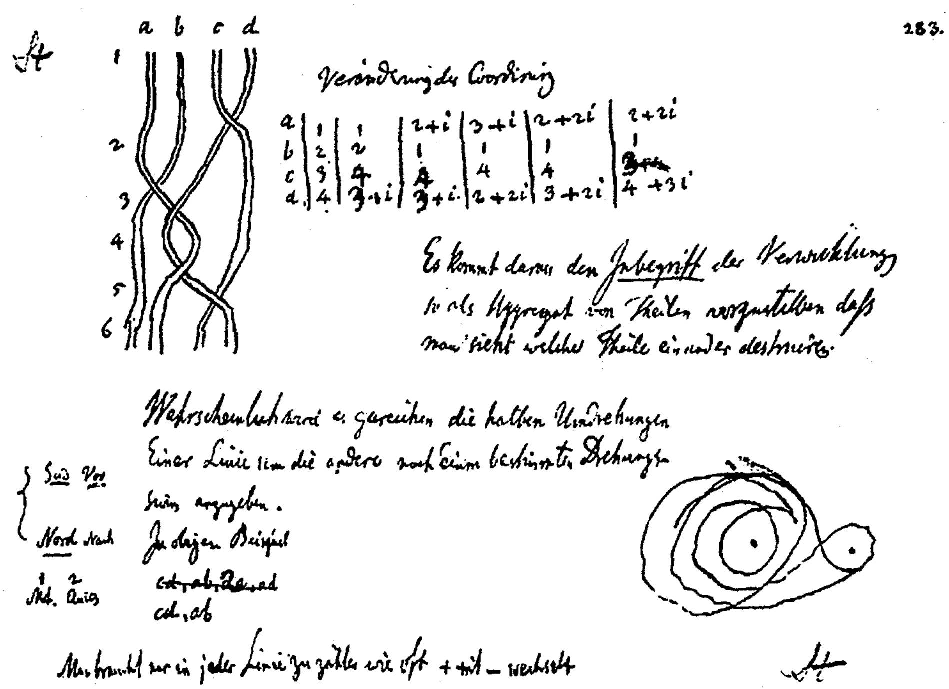

Figure 2. Page 283 of Gauss's Handbuch 7.

erator appears in the braid word). At this point, Gauss seems to have realized the necessity of finding an adequate way of keeping track of the orientations of the half-twists composing a braid. On the margin of the page, Gauss drew another sketch showing a curve winding around two points, probably illustrating the winding of a braid strand around two others as seen from above. Then follows the last sentence of the fragment:

One only has to count in every line how often + changes with −.

Given that Gauss did not specify what his signs + and − mean, the interpretation of this remark can only be tentative. The earlier mentioning of a "sense of rotation," the drawing on the right margin, and the notes "south before/north behind" on the left margin make it probable, however, that the signs do, indeed, refer to the two possible orientations of a half-twist. On this reading, Gauss proposed to count the difference between the numbers of positive and negative half-twists a given strand experiences in a possibly complex braid. For a two-strand braid, and on division by 2, this amounts to the modern combinatorial definition of the linking number.

The drawing on the margin and Gauss's knowledge of the index of plane curves make it probable that already at the time of writing, he was aware of an analytical method for the computation of the linking number, using the projection of a braid to a transversal plane: If the coordinates of two braid strands in $\mathbb{R}^3 \cong \mathbb{C}\oplus\mathbb{R}$ are given by $(z_1(t),t)$ and $(z_2(t),t)$, respectively, then their linking number is, of course, the (half-integer-valued) index of the curve $t \mapsto z_1(t) - z_2(t)$ with respect to zero. In this light, the fragment of 1833 would, indeed, be just another *computation* of the linking number, and not a *definition.*

The *second reading* of Gauss's conjecture (and the whole fragment) would be much stronger. On this reading, we would ascribe to Gauss the belief that in order to solve the classification problem for braids, it would possibly suffice to determine, in addition to the permutation associated with a braid, just the numbers of (signed) half-twists between all pairs of strands. The remarks following the conjecture do not contradict this stronger interpretation. Even though the conjecture is false, it would show a remarkable insight (and note that Gauss qualified his conjecture with the cautious "Probably . . .").

There is a sort of a middle course between the two readings discussed: While Gauss was looking for a notation for braids which enabled one to decide whether or not two braids were equivalent, he came close to defining a nontrivial invariant for braids, namely the last row of the table he set up (with a definite convention on crossings adopted). Allow me to sketch a corresponding elaboration of Gauss's ideas, freely using modern mathematical language. These variations on Gauss's theme are *not* intended to represent an historical reconstruction of his line of thought. At best, they might indicate the space of mathematical speculation in which Gauss was moving when he gave an afternoon's thought to the first braid.

A Gaussian Invariant of Braids

Let the n-strand braid group B_n be generated by $\sigma_1, \ldots, \sigma_{n-1}$, with defining relations

$$\sigma_k\sigma_l = \sigma_l\sigma_k \quad \text{if } |k - l| \geq 2;$$
$$\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1} \quad \text{for } k = 1, \ldots, n-2.$$

An action α of B_n on the lattice of Gaussian integers $\mathbb{Z}[i]$ may be defined by

$$\alpha(\sigma_k)(z) := \begin{cases} z, & \operatorname{Re} z \neq k, k+1 \\ z+1, & \operatorname{Re} z = k \\ z-1+i, & \operatorname{Re} z = k+1 \end{cases}$$
$$(k = 1, 2, \ldots, n-1).$$

The action of the inverse is given by

$$\alpha(\sigma_k^{-1})(z) := \begin{cases} z, & \operatorname{Re} z \neq k, k+1 \\ z+1-i, & \operatorname{Re} z = k \\ z-1, & \operatorname{Re} z = k+1. \end{cases}$$

One easily verifies the defining relations of B_n for this action.

Given a braid word w in the generators σ_k, we can consider the corresponding path of a point $z \in \mathbb{Z}[i]$. In Gauss's example, we have $w = \sigma_3\sigma_1\sigma_2^{-2}\sigma_3$, so for $z = 1$, we get the sequence

$$1,\ 1,\ 2,\ 3-i,\ 2-i,\ 2-i.$$

The paths of the n points $1, 2, \ldots, n \in \mathbb{Z}[i]$ determine the braid word completely, since in each step we can see which generator has been acting. In fact, we can take these paths as the entries of an improved version of Gauss's table, making each path one row of the table. In the example, the modified table reads

a	1	1	2	$3-i$	$2-i$	$2-i$
b	2	2	$1+i$	$1+i$	$1+i$	$1+i$
c	3	4	4	4	4	$3+i$
d	4	$3+i$	$3+i$	$2+i$	3	4

The columns of this table can easily be read off a diagram of the given braid: Just take the positions of the strands as real parts and at each crossing, add $\pm i$ to the *lower* strand according to the orientation of the crossing. By construction, the last column of the table is an easily calculated invariant of the given braid. It is determined by the permutation associated with w and, for each strand, the sum of the signs of its undercrossings. Of course, this invariant is not complete, as it is determined by the linking numbers between the strands of the braid.

The above construction may evidently be modified in various ways. For instance, we could consider a *symmetric* version of the action:

$$\beta(\sigma_k)(z) := \begin{cases} z, & \operatorname{Re} z \neq k, k+1 \\ z+1+i, & \operatorname{Re} z = k \\ z-1+i, & \operatorname{Re} z = k+1 \end{cases}$$
$$(k = 1, 2, \ldots, n-1).$$

For this action, the last column of the table for a braid word is of the form

$$\pi_w(1) + j(1)i, \quad \pi_w(2) + j(2)i, \ldots, \pi_w(n) + j(n)i,$$

where π_w is the permutation associated with a braid, and $j(k)$ is the sum of the signs of *all* crossings in which the kth strand is involved; in other words, $j(k)$ is twice the sum of the linking numbers between the kth strand and all other strands of the braid.

Very probably, the few lines in Gauss's notebook are the only trace of a mathematical activity in which one of the genuine objects of topology was first conceived. It would be difficult and unnecessary to decide which of the above readings comes closest to "what Gauss really did." Nevertheless, this fragment documents the invention of a new type of mathematical object, which would be called braids a century later. To be sure, there is no detailed discussion of the idea of composing braids, and, in any case, no explicit group-theoretical thinking. From the letter to Olbers written in 1802, we also know that Gauss had previously read Vandermonde's paper on weaving patterns. But still there is a certain acuity in Gauss's fragment which marks the borderline between vague ideas and a fairly definite object of thought.

The least we can say is that the fragment reveals that Gauss had an idea of how to describe the phenomenon of linking in braids by a number counting the signs of diagram crossings. This closes the gap in our interpretation of the fragment on the linking integral. We have thus obtained a rather clear picture of how this fragment was situated in Gauss's mathematical practice. Having first encountered the differential form dominating the mathematics of linking in the astronomical context of the *zodiacus* of asteroids, Gauss was led to recognize its topological content when he considered linked orbits of celestial bodies. Then, during his studies of the differential geometry of curved surfaces, motivated by geodetical work, he felt bound to reconsider topological issues. Among other things, he discussed the notion of an oriented area of closed curves in the plane or on the sphere, a topic which he most probably had considered already in the context of his investigation of the *zodiacus*. Around this time, he also gave some hours of thought to his braid and discovered the combinatorial way of determining the linking number. When he finally was led back to *geometria situs* in the context of his joint work with Weber on electromagnetic induction, it was easy for him to recall his earlier results. If we are to believe the indirect report by Möbius, Gauss renewed his earlier wish to write a treatise on linked (and perhaps knotted) space curves, but (as before) he was unable to produce something which satisfied his extraordinary publication standards. In his notebook, he made only this single mention, in a tone of resignation, and it gave the core result of Gauss's own contributions to the new discipline.

What is striking about this story are the roles of the various disciplines in the context of which the linking phenomenon was discussed. Although Gauss was, throughout his career, very clear that in the conceptual hierarchy of mathematics, *geometria situs* was fundamental, the practical motivation of dealing with linking and related topics came from the exact sciences of astronomy, geodesy, and electromagnetism. The mathematical activity of which the fragment of 1833 is an extremely condensed trace was one of *mathematization*, of a domain of very intuitive problems which had not yet been treated within the "sublime science of quantity," with the partial exception of those immediately related to complex integration. That these problems continued to reappear in several of the leading sciences of Gauss's day must be one of the reasons for the very high value he put on the slowly emerging science of topology.

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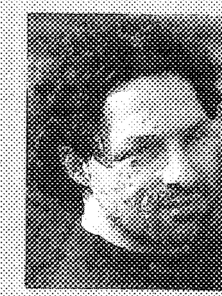

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