RAYLEIGH QUOTIENT ITERATION

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To help with the last part of Problem 5 in homework 2, here is a clarification of the convergence of Rayleigh Quotient Iteration (Theorem 27.3 of Trefethen & Bau, p. 208). We follow the notation of T&B throughout.

First assume that the normalized estimate $\boldsymbol{v}^{(k)}$ is close to the normalized eigenvector $\boldsymbol{q}_J$ with eigenvalue λ_J:

$$\|\boldsymbol{v}^{(k)} - \boldsymbol{q}_J\| = \varepsilon \tag{1}$$

for ε small. This means that

$$\boldsymbol{v}^{(k)} = C_{k,\varepsilon}\left(\boldsymbol{q}_J + \varepsilon\, \boldsymbol{u}^{(k)}\right), \tag{2}$$

with

$$C_{k,\varepsilon} = (1 + \varepsilon^2 \|\boldsymbol{u}^{(k)}\|^2)^{-1/2} \quad \text{and} \quad (\boldsymbol{q}_J)^T \boldsymbol{u}^{(k)} = 0. \tag{3}$$

The corresponding Rayleigh quotient estimate is then

$$\begin{aligned}
\lambda^{(k)} &= (\boldsymbol{v}^{(k)})^T A \boldsymbol{v}^{(k)} \\
&= (\boldsymbol{q}_J + \varepsilon\, \boldsymbol{u}^{(k)})^T A\, (\boldsymbol{q}_J + \varepsilon\, \boldsymbol{u}^{(k)}) + \mathrm{O}(\varepsilon^2) \\
&= \lambda_J + \varepsilon^2\, (\boldsymbol{u}^{(k)})^T A \boldsymbol{u}^{(k)} + \mathrm{O}(\varepsilon^2).
\end{aligned}$$

Hence, $|\lambda^{(k)} - \lambda_J| = \mathrm{O}(\varepsilon^2)$. Now take an inverse iteration step: we must solve

$$(A - \mu\, I)\, \boldsymbol{w} = \boldsymbol{v}^{(k)} = C_{k,\varepsilon}\left(\boldsymbol{q}_J + \varepsilon\, \boldsymbol{u}^{(k)}\right) \tag{4}$$

for $\boldsymbol{w}$. The solution is

$$\begin{aligned}
\boldsymbol{w} &= C_{k,\varepsilon}\, (A - \mu\, I)^{-1} (\boldsymbol{q}_J + \varepsilon\, \boldsymbol{u}^{(k)}) \\
&= C_{k,\varepsilon}\, (\lambda_J - \mu)^{-1} \boldsymbol{q}_J + C_{k,\varepsilon}\, \varepsilon (A - \mu\, I)^{-1} \boldsymbol{u}^{(k)}.
\end{aligned}$$

Hence,

$$C_{k,\varepsilon}^{-1}\, (\lambda_J - \mu) \boldsymbol{w} = \boldsymbol{q}_J + \varepsilon\, (\lambda_J - \mu)(A - \mu\, I)^{-1} \boldsymbol{u}^{(k)}. \tag{5}$$

Now letting $\mu = \lambda^{(k)}$, we have $\lambda_J - \mu = \lambda_J - \lambda^{(k)} = \mathrm{O}(\varepsilon^2)$, so that

$$\varepsilon\, (\lambda_J - \lambda^{(k)})(A - \lambda^{(k)}\, I)^{-1} \boldsymbol{u}^{(k)} =: \varepsilon^3\, \boldsymbol{u}^{(k+1)} = \mathrm{O}(\varepsilon^3). \tag{6}$$

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This is true as long as $(A - \lambda^{(k)}\, I)^{-1}\boldsymbol{u}^{(k)}$ doesn't blow up, but it doesn't since $\boldsymbol{u}^{(k)}$ is orthogonal to $\boldsymbol{q}_J$. Note also that $\boldsymbol{u}^{(k+1)}$ remains orthogonal to $\boldsymbol{q}_J$. Thus, to leading order in ε we get the improved estimate to $\boldsymbol{q}_J$ by normalizing $\boldsymbol{w}$:

$$\boldsymbol{v}^{(k+1)} = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = C_{k+1,\varepsilon^3}\left(\boldsymbol{q}_J + \varepsilon^3\,\boldsymbol{u}^{(k+1)}\right). \tag{7}$$

To get the next level of improvement, start again from (2) with ε replaced by ε^3 and k by $k+1$, yielding an $(\varepsilon^3)^3 = \varepsilon^9$ improvement for the estimate $\boldsymbol{v}^{(k+2)}$.