MATH 715 - Computational Mathematics II

HW #4

Due 11:00am, Thursday Apr. 12

1. Give a variational formulation of the problem

$$\frac{\partial^4 u}{\partial x^4} = f, \quad 0 < x < 1, \quad f \in L^2(0,1), \tag{1}$$

$$u(0) = u''(0) = u'(1) = u'''(1) = 0. \tag{2}$$

Prove that there exists a unique solution to the variational problem with $u \in H^2(0,1)$ (invoke the Lax-Milgram theorem).

2. Recall the biharmonic equation, $\Delta^2 u = f$ in $\Omega \subset \mathbb{R}^2$, with $u = \partial u/\partial n = 0$ on the boundary $\partial\Omega$, and $f \in L^2(\Omega)$. The variational form, as we derived in class, is given by

$$a(u,v) = \int_\Omega \Delta u \Delta v \, dS = (f,v) \qquad \forall v \in H_0^2(\Omega). \tag{3}$$

Prove that $\ell(v) = (f,v)$ is continuous for $v \in H_0^2(\Omega)$, and that $a(u,v)$ is continuous and coercive for $u,v \in H_0^2(\Omega)$. Lax-Milgram then assures us that there exists a unique solution to Eqn. (3).

3. Let K be a tetrahedron with vertices a^i, $i = 1,2,3,4$, and let a^{ij} be the midpoint on the line between a^i and a^j, $i < j$. Show that a function $v \in P_2(K)$ is uniquely determined by the degrees of freedom $v(a^i)$, $v(a^{ij})$, $i,j = 1,2,3,4$, $i < j$. Show that the corresponding finite element space V_h satisfies $V_h \subset C^0(\Omega)$.

4. Consider the Neumann problem

$$-\Delta u + u = f \quad \text{ in } \Omega, \tag{4}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega. \tag{5}$$

Assume Ω is the unit square $[0,1] \times [0,1]$.

(a) Derive the variational form of the PDE, using test functions $v \in H^1(\Omega)$.

(b) Using uniform square elements, formulate a finite element method using a piecewise bilinear approximation.

(c) Write a computer program in your favorite language to implement this method. Solve the problem with $f = \cos(\pi x) - \cos(\pi y)$ and $g = 0$ using 16, 64, and 256 elements. Plot the solution as a surface in each of these cases. Since the exact solution is known, you can compare your numerical solution with the exact solution to validate the accuracy (second order) of your scheme to see if the error reduces by a factor of 4 as you reduces the size of elements by half.

You may use a standard direct routine to solve $A\xi = b$ (i.e. Gaussian elimination). The simplest way for those without a favorite programming language is to use Matlab ($\xi = A\backslash b$).

(5) →

5. Solve numerically the following differential equation by the finite element method using piecewise linear elements:

$$-u_{xx} - u + x^2 = 0, \qquad 0 < x < 1, \tag{6}$$

for the following set of boundary conditions:

$$\text{(a) Dirichlet boundary conditions:} \qquad u(0) = u(1) = 0, \tag{7}$$

$$\text{(b) Mixed boundary conditions:} \qquad u(0) = 0,\ u_x(1) = 1, \tag{8}$$

$$\text{(c) Neumann boundary conditions:} \qquad u_x(0) = 1,\ u_x(1) = 4/3. \tag{9}$$

Note that for Neumann boundary conditions, none of the primary dependent variables is specified, and therefore the coefficient matrix remains unaltered. Use 20 linear finite elements, and output your numerical results in figures. Also determine $u_x(0)$. Discuss why you think your numerical results are correct.