MATH 715 - Computational Mathematics II

HW #3

Due 2:30pm, Thursday Mar. 22

1. Let $A \in \mathbb{C}^{m\times m}$ be arbitrary. The set of all Rayleigh quotients of A, corresponding to all nonzero vectors $x \in \mathbb{C}^m$, is known as the *field of values* or the *numerical range* of A, a subset of the complex plane denoted by $W(A)$.

(a) Show that $W(A)$ contains the convex hull of the eigenvalues of A.

(b) Show that if A is normal, then $W(A)$ is equal to the convex hull of the eigenvalues of A.

2. Let A be an $n \times n$ tridiagonal matrix with $a_{i,i+1} = a_{i+1,i} = -1$ and $a_{ii} = 3$, and let $b \in \mathbb{R}^n$. For which values of the parameter ω does the iteration

$$x_{k+1} = x_k + \omega(b - Ax_k), \quad k = 0, 1, 2, ... \tag{1}$$

converge to a solution of $Ax = b$ for any starting value $x_0 \in \mathbb{R}^n$? Test your result on a computer for $n = 5$ and comment on your findings.

3. Consider the real system of linear equations $Ax = b$, where A is nonsingular and satisfies $(x, Ax) > 0$ for all real $x \neq 0$, where $(x, y) = x^T y$ is the Euclidean inner product.

(a) Show that $(x, Ax) = (x, Mx)$ for all real x, where $M = (A + A^T)/2$ is the symmetric part of A.

(b) Prove that $(x, Ax)/(x, x) \geq \lambda_{\min}(M) > 0$, where $\lambda_{\min}(M)$ is the smallest eigenvalue of M.

(c) Consider the iterative sequence $x_{n+1} = x_n + \alpha_n r_n$, where $r_n = b - Ax_n$ is the residual, and α_n is chosen to minimize $\|r_{n+1}\|_2$ as a function of α_n. Prove that

$$\frac{\|r_{n+1}\|_2}{\|r_n\|_2} \leq \left(1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^T A)}\right)^{1/2}. \tag{2}$$

4. Let A be the 100×100 tridiagonal symmetric matrix with $1, 2, ..., 100$ on the diagonal and 1 on the sub- and super- diagonals, and set $b = (1, 1, ..., 1)^T$. Write a program that takes 100 steps of the conjugate gradient algorithm, and separately a program that takes 100 steps of the steepest descent algorithm, to approximately solve $Ax = b$. Produce a plot with four curves: the computed residual norms $\|r_n\|_2$ and the actual residual norms $\|b - Ax_n\|_2$ for CG, the residual norms $\|r_n\|_2$ for steepest descent, and the estimate $2(\sqrt{\kappa} - 1)^n/(\sqrt{\kappa} + 1)^n$. Comment on your results.

5. Prove that if w is continuous on $[0, 1]$, and

$$\int_0^1 wv\,dx = 0 \ \ \forall v \in V, \tag{3}$$

$$V = \{v \in C[0, 1], v_x \text{ piecewise continuous}, v(0) = v(1) = 0\}, \tag{4}$$

then $w(x) = 0$ for $x \in [0, 1]$.