MATH 715 - Computational Mathematics II

HW #2

Due 2:30pm, Thursday Mar. 1

1. Determine the (a) eigenvalues, (b) determinant, and (c) singular values of a Householder reflector. For the given eigenvalues, give a geometric argument as well as an algebraic proof.

2. (a) Write a numerical function $[W, R] = house(A)$ that computes an implicit representation of a full QR factorization $A = QR$ of an $m \times n$ matrix A with $m \geq n$ using Householder reflections. The output variables are a lower triangular matrix $W \in \mathbb{C}^{m\times n}$ whose columns are the vectors v_k defining the successive Householder reflections, and a triangular matrix $R \in \mathbb{C}^{n\times n}$.

(b) Write a numerical function $Q = formQ(W)$ that takes the matrix W produced above and generates a corresponding $m \times m$ orthogonal matrix Q.

3. Let Z be the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{pmatrix} \tag{1}$$

Compute the reduced QR factorizations of Z by the modified Gram Schmidt procedure from HW # 1, using the Householder approach from the previous problem, and using MATLAB's built-in command $[Q, R] = qr(Z, 0)$. Compare the results and comment on any differences. Which approach do you think MATLAB uses?

4. *Gerschgorin's circle theorem.* For any matrix $A \in \mathbb{R}^{m\times m}$, every eigenvalue of A lies in at least one one of the m circular disks in the complex plane with centers a_{ii} and radii $\sum_{i\neq j} |a_{ij}|$. Moreover, if n of these disks form a connected domain that is disjoint from the other $m - n$ disks, then there are precisely n eigenvalues of A within this domain.

(a) Prove the first part of Gerschgorin's theorem (hint: let λ be any eigenvalue of A and x a corresponding eigenvector with largest entry 1.)

(b) Prove the second part. (Hint: deform A to a diagonal matrix and use the fact that the eigenvalues of a matrix are continuous functions of its entries.)

(c) Give estimates based on Gerschgorin's theorem for the eigenvalues of

$$A = \begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \quad |\epsilon| < 1. \tag{2}$$

(d) Find a way to establish the tighter bound $|\lambda_3 - 1| \leq \epsilon^2$ on the smallest eigenvalue of A. (Hint: consider diagonal similarity transformations).

5. Show that for a nonhermitian matrix $A \in \mathbb{C}^{m \times m}$, the Rayleigh quotient $r(x)$ gives an eigenvalue estimate whose accuracy is generally linear, not quadratic. Explain what convergence rate this suggests for the Rayleigh quotient iteration applied to nonhermitian matrices.