Lecture 35: Applications of Laplace transform

$$L[f](s) = \tilde{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$L^{-1}[\tilde{f}](t) = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)e^{st}\,ds$$

example: solve $u_t + u_x = x, \quad x>0, \; t>0$

$$u(t,0) = 0, \quad u(0,x) = 0$$

$$\tilde{u}(s,x) = \int_0^\infty u(t,x)e^{-st}\,dt$$

$$\tilde{u}_t(s,x) = -u(0,x) + s\tilde{u}(s,x)$$

$\frac{1}{s} = L[1]$

$$s\tilde{u}(s,x) - \underbrace{u(0,x)}_{0} + \tilde{u}_x(s,x) = \frac{x}{s}$$

First-order ODE (Particular sol'n: $\frac{x}{s^2}$)

Find:

$$\tilde{u}(s,x) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{e^{-sx}}{s^3}$$

Now invert. Use

$$L[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}, \quad p > -1$$

$$L[t^n] = \frac{n!}{s^{n+1}}, \quad 0 < n \in \mathbb{Z}$$

So $$L^{-1}[s^{-n}] = \frac{1}{(n-1)!} t^{n-1}, \quad 0 < n \in \mathbb{Z}$$

Also, LTs satisfy a convolution theorem:

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$L[f * g](s) = L[f](s)\, L[g](s)$$

$\tilde{f} = e^{-sx}$ $\quad$ $\tilde{g} = \frac{1}{s^3}$

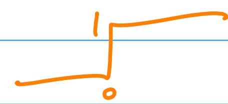
$f = \delta(t-x)$ $\quad$ $g = \frac{1}{2}t^2$

$$\mathcal{L}^{-1}\left[\frac{e^{-sx}}{s^3}\right](t) = (f*g)(t)$$

$$= \int_0^t \delta(\tau - x)\,\frac{1}{2}(t-\tau)^2\,d\tau$$

$$= \frac{1}{2}\,\sigma(t-x)\,(t-x)^2$$

step function



So finally:

$$\boxed{u(t,x) = xt - \frac{1}{2}t^2 + \sigma(t-x)\,(t-x)^2}$$

In general the inversion is the hardest step.

Luckily, if we want asymptotic results as $t \to \infty$, there are lots of theorems, most notably

Watson's Lemma: $f(t)$ integrable in $(0,b)$

with $$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n}, \quad t \to 0^+$$
$\alpha > -1$
$\beta > 0$

Then

$$\mathcal{L}[f](s) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha+\beta n+1)}{s^{\alpha+\beta n+1}}, \quad s \to \infty$$

Watson's Lemma tells us that the small t behavior of $f(t)$ is tied to the large s behavior of $\tilde{f}(s)$.

Conversely, for large t the ILT will be dominated by the singularity furthest to the right. (including branch points)