Lecture 34: Laplace transforms

Fourier transforms → solve BVP or periodic problems

Laplace transforms → solve IVP

$f(t)$ defined for $t \geq 0$

$$\underbrace{L[f](s)}_{\tilde{f}(s)} = \int_0^\infty f(t)\, e^{-st}\, dt$$

example: (i) $f(t) = 1$:

$$L[1](s) = \int_0^\infty e^{-st}\, dt = \frac{e^{-st}}{(-s)}\Bigg|_0^\infty = \frac{1}{s}$$

(ii) $f(t) = \delta(t-t_0)$:

$t_0 > 0$

$$\tilde{f}(s) = \int_0^\infty \delta(t-t_0)\, e^{-st}\, dt = e^{-st_0}$$

(iii) $f(t) = \sigma(t-t_0)$

$t_0 > 0$

$$\tilde{f}(s) = \int_{t_0}^\infty e^{-st}\, dt = \frac{e^{-st}}{(-s)}\Bigg|_{t_0}^\infty = \frac{e^{-st_0}}{s}$$

(iv) $f(t) = e^{-\gamma t}$

Re $s + \gamma > 0$

$$\tilde{f}(s) = \int_0^\infty e^{-(s+\gamma)t}\,dt = \left.\frac{e^{-(s+\gamma)t}}{-(s+\gamma)}\right|_0^\infty = \frac{1}{s+\gamma}$$

Derivative formula:

$$\mathcal{L}[f'](s) = \int_0^\infty f'(t)\,e^{-st}\,dt$$

$$= f(t)\,e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)\,e^{-st}\,dt$$

$$\boxed{\mathcal{L}[f'](s) = -f(0) + s\,\mathcal{L}[f](s)}$$

example: Solve $\frac{df}{dt} = g(t)$, $f(0) = f_0$

$$\mathcal{L}[f'] = \mathcal{L}[g]$$

$$-f(0) + s\,\mathcal{L}[f] = \mathcal{L}[g]$$

$$\mathcal{L}[f] = s^{-1}\left(\mathcal{L}[g] + f(0)\right)$$

$$f(t) = \mathcal{L}^{-1}\left[s^{-1}\left(\mathcal{L}[g](s) + f(0)\right]\right](t)$$

inverse LT!

Inverse Laplace transform:

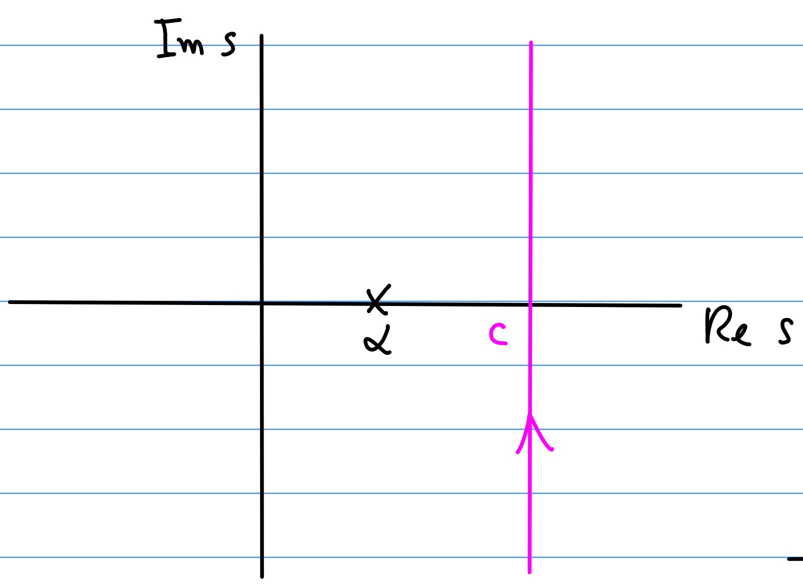
$$\mathcal{L}^{-1}[\tilde{f}](t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\, e^{st}\, ds$$

Bromwich integral

c is a real number chosen such that all the singularities of $\tilde{f}(s)$ are to the left of the contour.

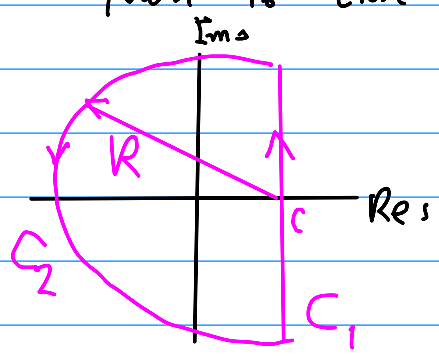
(no sing → Fourier trans.)

example: $\tilde{f}(s) = \dfrac{1}{s-\alpha}$



$$\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s-\alpha}\, ds$$

We want to use the residue theorem. Need to close contour.



Parametrize C_2:

$$s = c + Re^{i\theta}, \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

$$ds = iRe^{i\theta}\, d\theta$$

$$I_2 = \int_{C_2} \frac{e^{st}}{s-\alpha}\, ds = iR \int_{\pi/2}^{3\pi/2} \frac{e^{i\theta}\, e^{(c+Re^{i\theta})t}}{Re^{i\theta}+c-\alpha}\, d\theta$$

$$|I_2| \leq Re^{ct} \int_{\pi/2}^{3\pi/2} \frac{e^{2R\cos\theta\, t}}{(R^2+(c-\alpha)^2+4R(c-\alpha)\cos\theta)^{1/2}}\, d\theta$$

$\psi = \theta - \frac{\pi}{2}$

$$= 2Re^{ct} \int_{0}^{\pi/2} \frac{e^{-2R\sin\psi\, t}}{(R^2+(c-\alpha)^2-4R(c-\alpha)\sin\psi)^{1/2}}\, d\psi$$

$$\leq 2Re^{ct} \int_{0}^{\pi/2} \frac{e^{-2R\sin\psi\, t}}{R+c-\alpha}\, d\psi$$

Then use $\sin\psi \geq \frac{2}{\pi}\psi$, $0 \leq \psi \leq \pi/2$,

to show $I_2 \to 0$ as $R \to \infty$.

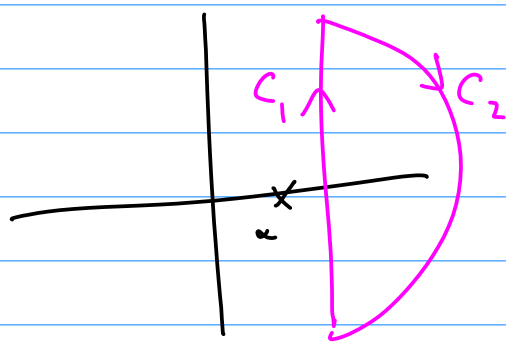
$$\int_{C_1} + \int_{C_2} \frac{e^{st}}{s-\alpha}\, ds = 2\pi i \,\{\text{sum of residues}\}$$
$$= 2\pi i \; e^{\alpha t}$$

But $\int_{C_2} \to 0$ as $R \to \infty$, so

$$f(t) = e^{\alpha t}, \quad t > 0$$

What about $t<0$? Then close contour on the right:

(so $\mathrm{Re}\, st < 0$)



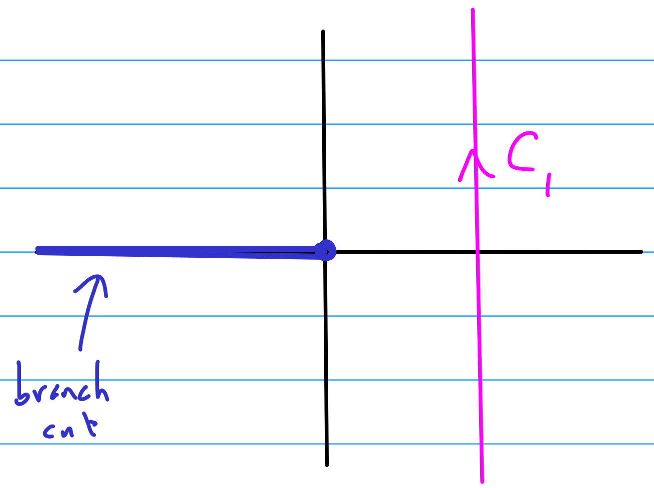
$\int_{C_1} \to 0$

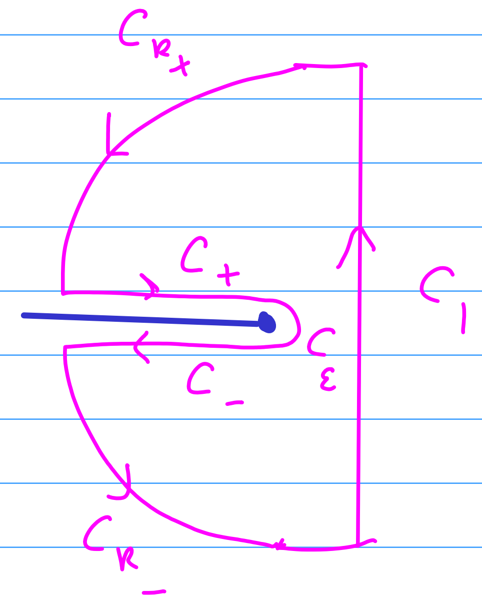
and $f(t) = 0, \quad t < 0.$

Conclude: inverse LT of $\frac{1}{s-\alpha}$ is

$$f(t) = \begin{cases} e^{\alpha t}, & t > 0 \\ 0, & t < 0 \end{cases}$$

example: inverse LT of $s^{-1/2}$:





On C_+: $\quad s \simeq s + i\varepsilon = |s| e^{i\pi}$

$$s^{-1/2} = |s|^{-1/2} e^{i\pi/2} = i\,|s|^{-1/2}$$

On C_-: $\quad s = s - i\varepsilon \simeq |s| e^{-i\pi}$

$$s^{-1/2} = |s|^{-1/2} e^{-i\pi/2} = -i\,|s|^{-3/2}$$

$$\int_{C_+} s^{-1/2} e^{st}\, ds = \int_0^\infty (i)\, \xi^{-1/2} e^{-\xi t} (-d\xi) = -i\sqrt{\frac{\pi}{t}}$$

$$\int_{C_-} s^{-1/2} e^{st}\, ds = \int_0^\infty (-i)\, \xi^{-1/2} e^{-\xi t} (d\xi) = -i\sqrt{\frac{\pi}{t}}$$

$s = \varepsilon e^{i\theta}, \quad ds = \varepsilon i \, d\theta$

$$\int_{C_\varepsilon} s^{-1/2} e^{st} \, ds = \int_{\theta = \pi/2}^{-\pi/2} e^{-i\theta/2} \varepsilon^{-1/2} e^{\varepsilon e^{i\theta}} \varepsilon i \, d\theta$$

$\to 0$ as $\varepsilon \to 0$

$$\int_{C_1 + C_{R_+} + C_+ + C_\varepsilon + C_- + C_{R_-}} = 0$$

$0 \quad 0 \quad 0$

$$\int_{C_1} = - \int_{C_+ + C_-} = 2i \sqrt{\frac{\pi}{t}}$$

Conclude,

$$L^{-1}\left[s^{-1/2}\right] = \begin{cases} \dfrac{1}{\sqrt{\pi t}}, & t > 0 \\ 0, & t < 0 \end{cases}$$

We can also check directly:

$$L\left[t^{-1/2}\right] = \int_0^\infty t^{-1/2} e^{-st} \, dt = \sqrt{\frac{\pi}{s}}$$