Lecture 28: Multiscale analysis

I. MULTISCALE EXPANSION AND HOMOGENIZATION

We start with the advection–diffusion equation for the concentration $\varphi(t,\boldsymbol{r})$ of some quantity,

$$\partial_t\varphi(t,\boldsymbol{r}) + \boldsymbol{u}(\boldsymbol{r})\cdot\nabla_{\boldsymbol{r}}\varphi(t,\boldsymbol{r}) = D\Delta_{\boldsymbol{r}}\varphi(t,\boldsymbol{r}) \tag{1}$$

with $\nabla_{\boldsymbol{r}}\cdot\boldsymbol{u} = 0$. For simplicity we take $\boldsymbol{u}$ to be a function of space only. Assume a typical lengthscale of $\boldsymbol{u}$ is ℓ, and that the initial condition for φ varies on a scale L that is large with respect to ℓ. Define $\varepsilon = \ell/L \ll 1$. We write $\varphi(0,\boldsymbol{r}) = \varphi_0(\varepsilon\,\boldsymbol{r})$.

Now introduce the large scale and slow time, whose magnitudes are related to the fast variables by

$$\boldsymbol{R} \sim \varepsilon\,\boldsymbol{r}, \qquad T \sim \varepsilon^2 t\,, \tag{2}$$

and assume that the concentration depends on these scales,

$$\varphi(t,\boldsymbol{r}) = \varphi^\varepsilon(T,\boldsymbol{r},\boldsymbol{R}). \tag{3}$$

Using $\partial_t \to \varepsilon^2\,\partial_T$, $\nabla_{\boldsymbol{r}} \to \nabla_{\boldsymbol{r}} + \varepsilon\,\nabla_{\boldsymbol{R}}$, Eq. (1) becomes

$$\mathcal{L}\varphi^\varepsilon + \varepsilon^2\,\partial_T\varphi^\varepsilon + \varepsilon\,\boldsymbol{u}(\boldsymbol{r})\cdot\nabla_{\boldsymbol{R}}\varphi^\varepsilon = 2\varepsilon\,D\nabla_{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{R}}\varphi^\varepsilon + \varepsilon^2 D\Delta_{\boldsymbol{R}}\varphi^\varepsilon \tag{4}$$

where the velocity field is assumed to only depend on the short lengthscale $\boldsymbol{r}$, and we have defined the linear operator

$$\mathcal{L} := -D\Delta_{\boldsymbol{r}} + \boldsymbol{u}\cdot\nabla_{\boldsymbol{r}}\,. \tag{5}$$

We expand the concentration in a power series in ε,

$$\varphi^\varepsilon(T,\boldsymbol{r},\boldsymbol{R}) = \varphi^{(0)}(T,\boldsymbol{r},\boldsymbol{R}) + \varepsilon\,\varphi^{(1)}(T,\boldsymbol{r},\boldsymbol{R}) + \ldots \tag{6}$$

and at order ε^0 obtain from Eq. (4),

$$\mathcal{L}\varphi^{(0)} = 0. \tag{7}$$

So far we have not discussed boundary conditions: we will assume that φ^ε is periodic in $\boldsymbol{r}$. The advection–diffusion operator is a second-order parabolic operator, and it obeys a weak maximum principle (see Evans,[1] p. 389). The solution to (7) must thus achieve its maximum and minimum on a boundary. Since the boundary conditions are periodic, there is no boundary, and so the only solution to (7) is a constant in $\boldsymbol{r}$, that is $\varphi^{(0)}(T,\boldsymbol{r},\boldsymbol{R}) = \Phi(T,\boldsymbol{R})$.

At order ε^1, Eq. (4) with the expansion (6) gives

$$\mathcal{L}\varphi^{(1)} + \boldsymbol{u}\cdot\nabla_{\boldsymbol{R}}\Phi = 0. \tag{8}$$

If there are to be solutions to the linear system, the Fredholm alternative must be satisfied. With respect to the standard inner product,

$$\langle f, g\rangle := \frac{1}{V}\int_\Omega f\,g\,\mathrm{d}^3r, \qquad V := \int_\Omega \mathrm{d}^3 r, \tag{9}$$

the adjoint operator to (5) is

$$\mathcal{L}^* := -D\Delta_{\boldsymbol{r}} - \boldsymbol{u}\cdot\nabla_{\boldsymbol{r}}\,, \tag{10}$$

assuming appropriate boundary conditions (periodic in $\boldsymbol{r}$ in our case). As for (7), the nontrivial solutions to $\mathcal{L}^* v = 0$ are $v =$ constant (take $v = 1$). The Fredholm alternative for Eq. (8) is then obtained from $\left\langle 1, \mathcal{L}\varphi^{(1)}\right\rangle = 0$, which gives

$$\langle 1, \boldsymbol{u}\rangle \cdot \nabla_{\boldsymbol{R}}\Phi = 0 \tag{11}$$

which is satisfied for $\langle 1, \boldsymbol{u}\rangle = 0$, i.e., the velocity field has zero spatial average.

From Eqs. (8) and (11) we must solve

$$\mathcal{L}\varphi^{(1)} + \boldsymbol{u}\cdot\nabla_{\boldsymbol{R}}\Phi = 0\,. \tag{12}$$

The solution to this is $\varphi^{(1)} = \boldsymbol{\chi}(\boldsymbol{r})\cdot\nabla_{\boldsymbol{R}}\Phi$, where

$$\mathcal{L}\boldsymbol{\chi} + \boldsymbol{u} = 0\,, \tag{13}$$

the so-called *cell problem*. Note that we must have $\langle 1, \mathcal{L}\boldsymbol{\chi}\rangle = 0$ for the cell problem to have a solution, and that $\boldsymbol{\chi}$ is not unique since we can add a constant to it. Without loss of generality, choose $\langle 1, \boldsymbol{\chi}\rangle = 0$.

Assuming the cell problem (13) has been solved, we can proceed to order ε^2 in Eq. (4),

$$\mathcal{L}\varphi^{(2)} + \partial_T\Phi + \boldsymbol{u}\cdot\nabla_{\boldsymbol{R}}\varphi^{(1)} = 2D\nabla_{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{R}}\varphi^{(1)} + D\Delta_{\boldsymbol{R}}\Phi\,. \tag{14}$$

Applying the Fredholm alternative to (14) and using $\left\langle 1, \mathcal{L}\varphi^{(2)}\right\rangle = 0$, we find

$$\partial_T\Phi + \nabla_{\boldsymbol{R}}\cdot(\langle 1, \boldsymbol{u}\,\boldsymbol{\chi}\rangle\cdot\nabla_{\boldsymbol{R}}\Phi) = 2D\nabla_{\boldsymbol{R}}\cdot(\langle 1, \nabla_{\boldsymbol{r}}\boldsymbol{\chi}\rangle\cdot\nabla_{\boldsymbol{R}}\Phi) + D\Delta_{\boldsymbol{R}}\Phi\,. \tag{15}$$

The average $\langle 1, \nabla_{\boldsymbol{r}}\boldsymbol{\chi}\rangle$ vanishes, and we thus finally obtain the homogenized diffusion equation

$$\partial_T\Phi = \nabla_{\boldsymbol{R}}\cdot(\mathbb{D}_{\text{eff}}\cdot\nabla_{\boldsymbol{R}}\Phi) \tag{16}$$

where the *effective diffusivity tensor* is

$$\mathbb{D}_{\text{eff}} := D\,\mathbb{I} - \langle \boldsymbol{u}, \boldsymbol{\chi}\rangle\ . \tag{17}$$

II. AN EXAMPLE

Consider the streamfunction for the *cellular flow*

$$\psi(x, y) = \sqrt{2}\,(U\ell/2\pi)\sin(2\pi x/\ell)\sin(2\pi y/\ell), \tag{18}$$

with velocity

$$\begin{aligned} u(x, y) &= \partial_y\psi = \sqrt{2}\,U\sin(2\pi x/\ell)\cos(2\pi y/\ell), \\ v(x, y) &= -\partial_x\psi = -\sqrt{2}\,U\cos(2\pi x/\ell)\sin(2\pi y/\ell). \end{aligned} \tag{19}$$

To compute the effective diffusivity, we need to solve the cell problem (13). Consider the ratio

$$\frac{|\boldsymbol{u}\cdot\nabla\boldsymbol{\chi}|}{|D\Delta\boldsymbol{\chi}|} \sim \frac{U\ell}{D} =: \text{Pe}, \tag{20}$$

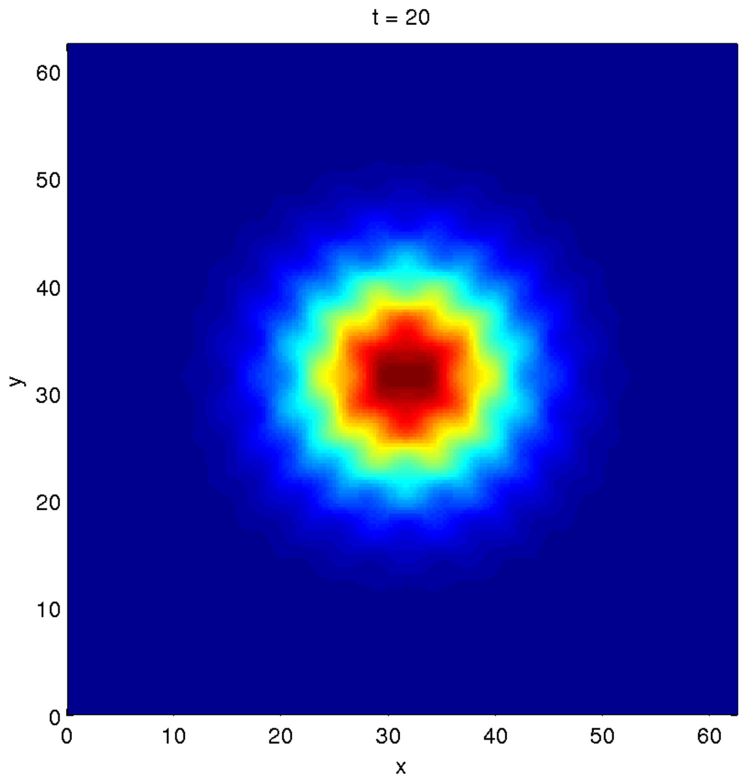

FIG. 1. Concentration field at $t = 20$ for $U = 1$, $\ell = 2\pi$, $D = 1$.

where Pe is the *Péclet number*. If the Péclet number is small, we can neglect the advection term in the cell problem, and get the simplified equation $D\Delta\boldsymbol{\chi} = \boldsymbol{u}$, or

$$D\Delta\chi_x = \sqrt{2}\,U\sin(2\pi x/\ell)\cos(2\pi y/\ell), \qquad D\Delta\chi_y = -\sqrt{2}\,U\cos(2\pi x/\ell)\sin(2\pi y/\ell), \tag{21}$$

with solution

$$\boldsymbol{\chi} = -\frac{\ell^2}{9\pi^2 D}\,\boldsymbol{u}. \tag{22}$$

We can then easily compute the effective diffusivity tensor by using $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = \frac{1}{2}U^2\mathbb{I}$ in (17):

$$\mathbb{D}_{\text{eff}} := D\left(1 + \frac{1}{16\pi^2}\,\text{Pe}^2\right)\mathbb{I}. \tag{23}$$

Figure 1 shows the concentration field for a numerical simulation at small Pe. In Figure 2 we compare the evolution of the variance to that implied by (23). Note that there is a short transient, since the initial condition has a small scale and so must spread out before scale separation is achieved.

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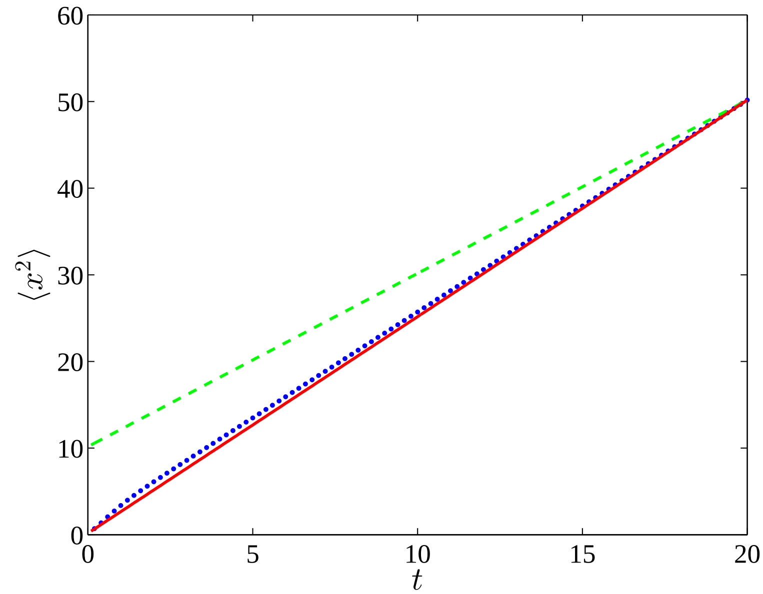

FIG. 2. Evolution of variance for $U = 1$, $\ell = 2\pi$, $D = 1$. The dots are numerical simulations, the green dashed line is $2Dt$, and the red line is $2\mathbb{D}_{\text{eff}}t$, where $\mathbb{D}_{\text{eff}}$ is defined in (23).

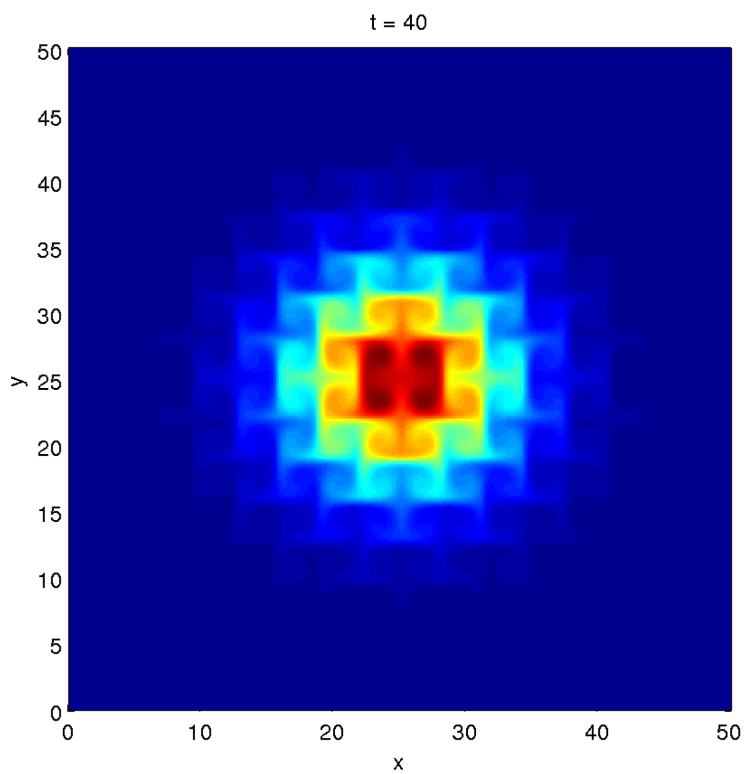

FIG. 3. Concentration field at $t = 40$ for $U = 1$, $\ell = 2\pi$, $D = 0.1$.

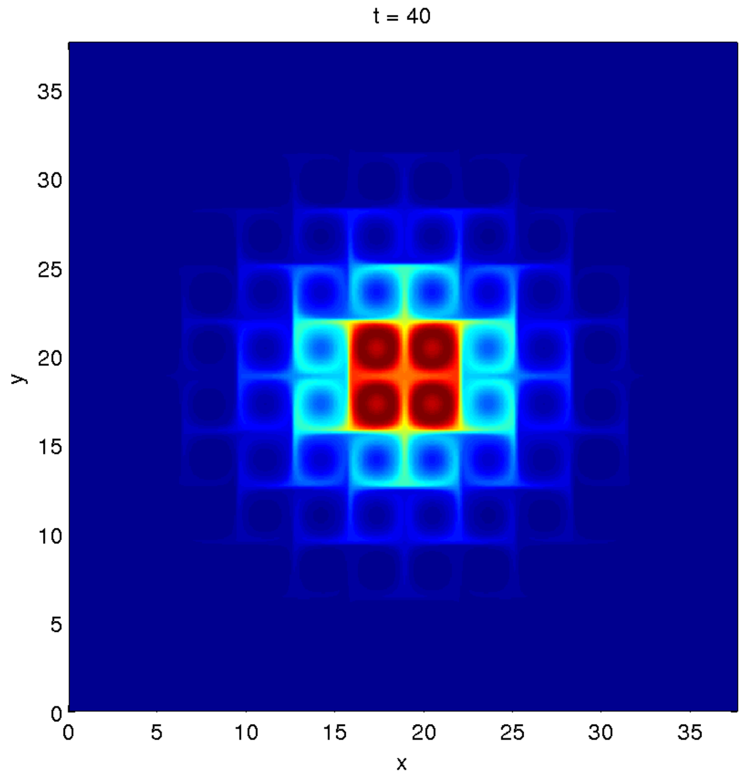

FIG. 4. Concentration field at $t = 40$ for $U = 1$, $\ell = 2\pi$, $D = 0.01$.

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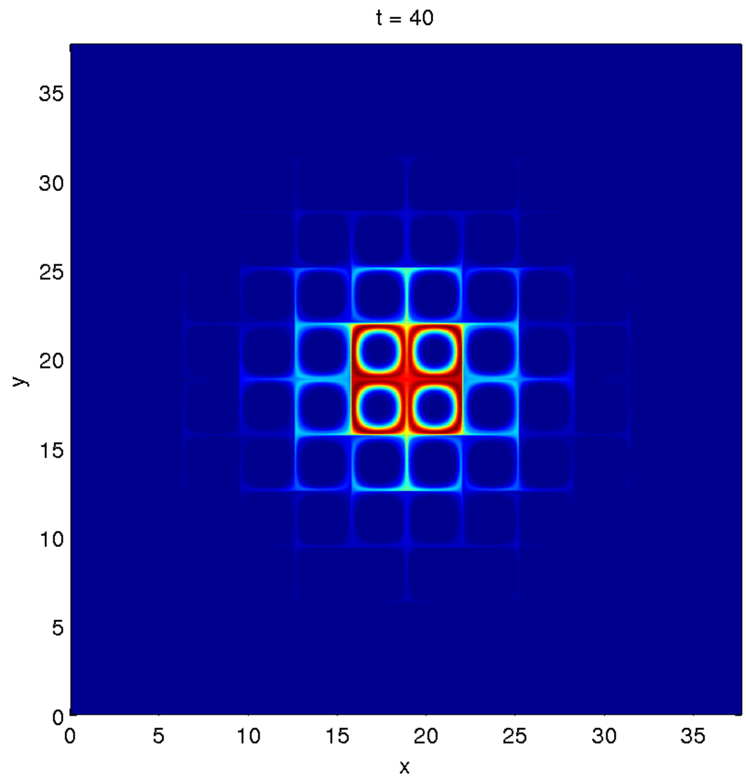

FIG. 5. Concentration field at $t = 40$ for $U = 1$, $\ell = 2\pi$, $D = 0.001$.