Lecture 27: Operator theory

Abstract setting: $L : U \to V$

L linear differential operator

U, V vector spaces

$\langle u, \tilde{u} \rangle$ inner product in U

$\langle\langle v, \tilde{v} \rangle\rangle$ inner product in V

Adjoint $L^* : V \to U$

$$\langle\langle L[u], v \rangle\rangle = \langle u, L^*[v] \rangle$$

$$\forall u \in U, v \in V$$

example: if L is an $m \times n$ matrix,

$$\langle u, L^*[v] \rangle = u_i (L^*_{ij} v_j)$$

$$\langle\langle L[u], v \rangle\rangle = (L_{ij} u_j) v_i = u_i L_{ji} v_j$$

$$\Rightarrow L^*_{ij} = L_{ji} \quad \text{matrix transpose}$$

example: $\langle u, \tilde{u} \rangle = \int_a^b u(x)\, \tilde{u}(x)\, dx$

$$\langle\langle v, \tilde{v} \rangle\rangle = \langle v, \tilde{v} \rangle$$

L^2 inner product (Hilbert space)

$$L[u] = D[u] = \frac{du}{dx}$$

$$\langle u, D^*[v] \rangle = \int_a^b u\, D^*[v]\, dx$$

$$\langle\langle D[u], v \rangle\rangle = \langle\langle u', v \rangle\rangle = \int_a^b u' v\, dx$$

$$= [uv]_a^b - \int_a^b u v'\, dx$$

If $[uv]_a^b$ vanishes, then $D^*[v] = -v'(x)$.

Need: $u(b)\, v(b) - u(a)\, v(a) = 0$

This could be satisfied in many ways.

$U = \{ u(x) \mid u(a) = u(b) = 0 \}$ Dirichlet (no condition on V)

or

$V = \{ v(x) \mid v(a) = v(b) = 0 \}$ Condition on V imposes condition on U so $D: U \to V$

$U = \{ u(x) \mid u'(a) = u'(b) = 0 \}$

Some basic properties:

- $(L^*)^* = L$
- $L: U \to V$, $M: V \to W$

$$(M \circ L)^* = L^* \circ M^* : W \to U$$

example: $D^2 u = u''$

$$(D^2)^* v = D^* \circ D^* v = v''$$

D^2 is self-adjoint.

(Careful about BCs!)

example: $\langle u, \tilde{u} \rangle = \iint_\Omega u(x,y)\, \tilde{u}(x,y)\, dx\, dy$

$$L = \nabla : \underset{\text{scalar field}}{U} \to \underset{\text{vector field}}{V}$$

$$\langle\langle \underline{v}, \underline{\tilde{v}} \rangle\rangle = \iint_\Omega \underline{v}(x,y) \cdot \underline{\tilde{v}}(x,y)\, dx\, dy$$

$$\langle u, L^*[\underline{v}] \rangle = \iint_\Omega u \, L^*[\underline{v}] \, dx \, dy$$

$$\langle\langle L[u], \underline{v} \rangle\rangle = \iint_\Omega \nabla u \cdot \underline{v} \, dx \, dy$$

$$= \oint_{\partial\Omega} (u\underline{v}) \cdot \underline{n} \, dS - \iint_\Omega u (\nabla \cdot \underline{v}) \, dx \, dy$$

For the boundary term to vanish, take either

(i) $u = 0$ on $\partial\Omega$

(ii) $\underline{v} \cdot \underline{n} = 0$ on $\partial\Omega$ (also $\underline{n} \cdot \nabla u = 0$ on $\partial\Omega$)

Then $\nabla^* \underline{v} = - \nabla \cdot \underline{v}$.

Crucially, for $L = \Delta = \nabla \cdot \nabla$

$$L^* = \Delta^* = \Delta$$

Laplacian is self-adjoint.

Fredholm alternative:

Solve $\quad L[u] = f$

$$\operatorname{coker} L = \ker L^* = \{ v \in V \mid L^*[v] = 0 \}$$

If $\; L[u] = f \;$ has a solution, then

$$\langle\langle v, f \rangle\rangle = 0 \quad \forall v \in \operatorname{coker} L$$

Proof: $\quad \langle\langle v, L[u] \rangle\rangle = \langle\langle v, f \rangle\rangle$

$$\langle \underbrace{L^*[v]}_{0}, u \rangle = \langle\langle v, f \rangle\rangle$$