

Lecture 27: Operator theory

Abstract setting: $L: U \rightarrow V$

L linear differential operator

U, V vector spaces

$\langle u, \tilde{u} \rangle$ inner product in U

$\langle\langle v, \tilde{v} \rangle\rangle$ inner product in V

Adjoint $L^*: V \rightarrow U$

$$\langle\langle L[u], v \rangle\rangle = \langle u, L^*[v] \rangle$$

$$\forall u \in U, v \in V$$

example: if L is an $m \times n$ matrix,

$$\langle u, L^*[v] \rangle = u_i (L^*_{ij} v_j)$$

$$\langle\langle L[u], v \rangle\rangle = (L_{ij} u_j) v_i = u_i L_{ji} v_j$$

$$\Rightarrow L^*_{ij} = L_{ji} \quad \underline{\text{matrix transpose}}$$

example: $\langle u, \tilde{u} \rangle = \int_a^b u(x) \tilde{u}(x) dx$

$$\langle\langle \nu, \tilde{\nu} \rangle\rangle = \langle \nu, \tilde{\nu} \rangle$$

L^2 inner product
(Hilbert space)

$$L[u] = D[u] = \frac{du}{dx}$$

$$\langle u, D^*[v] \rangle = \int_a^b u D^*[v] dx$$

$$\begin{aligned} \langle\langle D[u], v \rangle\rangle &= \langle\langle u', v \rangle\rangle = \int_a^b u' v dx \\ &= [uv]_a^b - \int_a^b u v' dx \end{aligned}$$

If $[uv]_a^b$ vanishes, then $D^*[v] = -v'(x)$.

Need: $u(b)v(b) - u(a)v(a) = 0$

This could be satisfied in many ways.

$$U = \{ u(x) \mid u(a) = u(b) = 0 \}$$

or

$$V = \{ v(x) \mid v(a) = v(b) = 0 \}$$

$$U = \{ u(x) \mid u'(a) = u'(b) = 0 \}$$

Dirichlet
(no. condition on V)

Condition on V
imposes condition on
 U so $D: U \rightarrow V$

Some basic properties:

$$\bullet (L^*)^* = L$$

$$\bullet L: U \rightarrow V, M: V \rightarrow W$$

$$(M \circ L)^* = L^* \circ M^* : W \rightarrow U$$

example: $D^2 u = u''$

$$(D^2)^* \nu = D^* \circ D^* \nu = \nu''$$

D^2 is self-adjoint.

(Careful about BCs!)

example: $\langle u, \tilde{u} \rangle = \iint_{\Omega} u(x,y) \tilde{u}(x,y) dx dy$

$$L = \nabla : U \rightarrow V$$

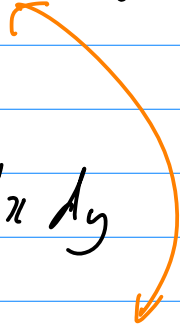
scalar
field

vector
field

$$\langle \underline{\nu}, \underline{\tilde{\nu}} \rangle = \iint_{\Omega} \underline{\nu}(x,y) \cdot \underline{\tilde{\nu}}(x,y) dx dy$$

$$\langle u, L^*[\underline{v}] \rangle = \iint_{\Omega} u L^*[\underline{v}] dx dy$$

$$\langle L[u], \underline{v} \rangle = \iint_{\Omega} \nabla u \cdot \underline{v} dx dy$$

$$= \oint_{\partial\Omega} (u \underline{v}) \cdot \underline{n} dS - \iint_{\Omega} u (\nabla \cdot \underline{v}) dx dy$$


For the boundary term to vanish, take either

(i) $u = 0$ on $\partial\Omega$

(ii) $\underline{v} \cdot \underline{n} = 0$ on $\partial\Omega$ (also $\underline{n} \cdot \nabla u = 0$ on $\partial\Omega$)

Then $\nabla^* \underline{v} = -\nabla \cdot \underline{v}$.

Crucially, for $L = \Delta = \nabla \cdot \nabla$

$$L^* = \Delta^* = \Delta$$

Laplacian is self-adjoint.

Fredholm alternative:

$$\text{Solve } L[u] = f$$

$$\text{coker } L = \text{ker } L^* = \{v \in V \mid L^*[v] = 0\}$$

If $L[u] = f$ has a solution, then

$$\langle\langle v, f \rangle\rangle = 0 \quad \forall v \in \text{coker } L$$

Proof: $\langle\langle v, L[u] \rangle\rangle = \langle\langle v, f \rangle\rangle$

$$\langle L^*[v], u \rangle = \langle\langle v, f \rangle\rangle$$

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