# Lecture 26: Solitons

Korteweg-de Vries equation (KdV)

$$u_t + u_{xxx} + uu_x = 0$$

surface wave in shallow water

Traveling wave solution: $u = v(\xi)$, $\xi = x - ct$

$$v''' + vv' - cv' = 0$$

Look for localized solutions: $v, v', v'' \to 0$ at $\pm\infty$.

$$v'' + \frac{1}{2}v^2 - cv = 0$$

← localized solution

$$v'v'' + \frac{1}{2}v'v^2 - cv'v = 0$$

Take $v > 0$.

$$\frac{1}{2}(v')^2 + \frac{1}{6}v^3 - \frac{1}{2}cv^2 = 0$$

Not quite right in Olver. Need $\pm$ otherwise $v' > 0$.

$$v'(\xi) = \pm v\sqrt{c - \frac{1}{3}v}$$

$$\text{LHS} = \pm\int \frac{dv}{v\sqrt{c - \frac{1}{3}v}} = \pm\left(-\frac{2}{\sqrt{c}} \operatorname{arctanh}\sqrt{1 - \frac{v}{3c}}\right)$$

$$= \xi + \delta$$

$\xi \to \infty$, $v \to 0^+$: $\quad LHS \to \pm\left(-\frac{2}{\sqrt{c}}\,\overbrace{\operatorname{arctanh}(1^-)}^{+\infty}\right) = +\infty$

So take "$-$" solution as $\xi \to \infty$

$\xi \to -\infty$, $v \to 0^+$: $\quad LHS \to \pm\left(-\frac{2}{\sqrt{c}}\,\overbrace{\operatorname{arctanh}(1^-)}^{+\infty}\right) = -\infty$

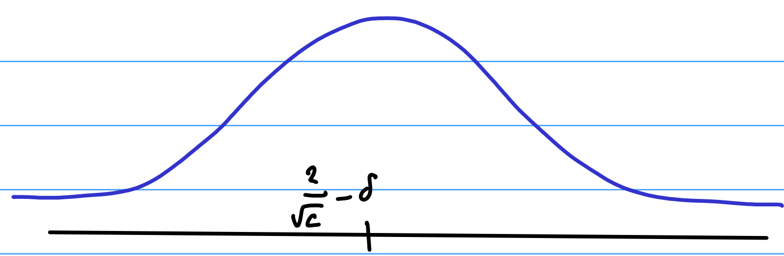
So take "$+$" solution as $\xi \to -\infty$.

The two branches meet at $v' = 0 \Rightarrow c = \frac{1}{3} v$.

$$\boxed{v(\xi) = 3c\,\operatorname{sech}^2\left(\tfrac{1}{2}\sqrt{c}\,\xi + \delta\right)}$$

solitary wave (soliton in this case)

Balance of dispersion and nonlinearity




$c > 0$, so moves to the right (left for $c < 0$)

Is this $C^\infty$ at $\xi = \frac{2}{\sqrt{c}} - \delta$? $\quad (v = 3c)$

$$(\log v)' = \pm\sqrt{c - \tfrac{1}{3} v}$$

$$(\log v)'' = \pm\frac{1}{2}\left(c - \tfrac{1}{3}v\right)^{-1/2}\left(-\tfrac{1}{3}v'\right)$$

$$= \pm\frac{1}{2}\left(c - \tfrac{1}{3}v\right)^{-1/2}\left(-\pm\tfrac{1}{3}v\sqrt{c - \tfrac{1}{3}v}\right)$$

$$= -\frac{1}{6}v$$

So now $(\log v)^{(n+2)} = -\frac{1}{6} v^{(n)}$.

Hence, if $v, v'$ continuous at max, so are all other derivatives.

Equations such as KdV are examples of integrable systems

These are nonlinear equations that admit full solutions in terms of solitons (+ extra).

Multi-soliton solutions exist; interaction still a topic of study (esp. in higher dim.)