Lecture 25: Dispersion

Beyond second order: 3rd order $\Rightarrow$ dispersion

$$u_t + u_{xxx} = 0 \quad , \quad u(0,x) = f(x) \qquad x \in \mathbb{R}$$

Fourier transform solution: $\hat{u}(t,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t,x) e^{-ikx} \, dx$

$$\hat{u}_t + (ik)^3 \hat{u} = 0$$

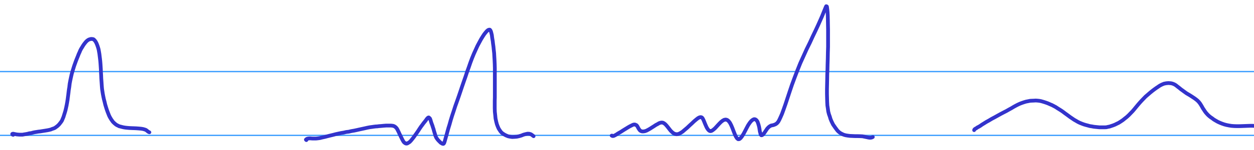
$$\hat{u}_t - ik^3 \hat{u} = 0 \qquad \hat{u} = \hat{f}(k) e^{ik^3 t}$$

$$\boxed{u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) \, e^{i(kx + k^3 t)} \, dk}$$

For example, for $f(x) = e^{-x^2}$, $\hat{f}(k) = \frac{e^{-k^2/4}}{\sqrt{2}}$,

$$u(t,x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i(kx + k^3) - \frac{k^2}{4}} \, dk$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2/4} \cos(kx + k^3 t) \, dk$$



Rapid oscillations on left propagate away and spread.
(dispersion)

Fundamental solution: $f(x) = \delta(x)$, $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}$

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(kx+k^3t)}\,dk = \frac{1}{\pi}\int_0^{\infty} \cos(kx+k^3t)\,dk$$

Convergence ?!

(maxima?)

Note that

$$\int_0^{\ell} \cos(kx+k^3t)\,dk = \int_0^{\ell} \frac{1}{x+3k^2t}\,\frac{d}{dk}\sin(kx+k^3t)\,dk$$

$$= \frac{\sin(kx+k^3t)}{x+3k^2t}\Bigg|_0^{\ell} + \int_0^{\ell} \frac{6kt\,\sin(kx+k^3t)}{(x+3k^2t)^2}\,dk$$

$\downarrow$ 0 as $\ell \to \infty$ — converges as $\ell \to \infty$

So the rapid oscillations make the integral convergent!

$u(t,x)$ not given analytically, but in terms of

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(sz + \frac{1}{3}s^3\right) ds$$ Airy function

$$u(t,x) = \frac{1}{(3t)^{1/3}} \mathrm{Ai}\left(\frac{x}{(3t)^{1/3}}\right)$$ $\sim t^{-1/3}$

Converges weakly to a δ function as $t \to 0$.

$$F(t,x;\xi) = \frac{1}{(3t)^{1/3}} \mathrm{Ai}\left(\frac{x-\xi}{(3t)^{1/3}}\right)$$ translation invariance

Need better intuition... so back to start $u_t + u_{xxx} = 0$

Linear, const. coeffs, so look for solutions $e^{i(kx-\omega t)}$

$$\omega = -k^3$$ dispersion relation

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-c_p t)}, \quad c_p = \frac{\omega}{k}$$

phase velocity

For $\omega = ck$, c const., $u_t + cu_x = 0$
transport, no dispersion

In general,

$$u(t,x) = \int_{-\infty}^{\infty} e^{i(kx - \omega t)} g(k)\, dk$$

$$\omega = \omega(k)$$

Moving frame:

$$u(t, \xi + ct) = \int_{-\infty}^{\infty} e^{i\varphi(k)t} h(k)\, dk = H(t)$$

where $\varphi(k) = ck - \omega(k)$, $h(k) = e^{ik\xi} g(k)$

Oscillatory integral $\to 0$ as $t \to \infty$.

Dominant behavior for large t? Stationary phase

$$\varphi'(k) = 0$$

$$\varphi'(k) = \omega'(k) - c = 0 \qquad c_g = \omega'(k)$$ group velocity

For $\omega = k^3$, we have $c_p = \frac{\omega}{k} = -k^2$

$c_g = \omega' = -3k^2$ faster