Lecture 24: Burgers' equation

$$u_t + u u_x = \gamma u_{xx}$$

1-dimensional fluid
no pressure

"Diffusive" when u, u_x are small

hyperbolic when large (shocks, etc.)

$$u(0,x) = f(x), \quad -\infty < x < \infty$$

Seek traveling wave solutions: $u(t,x) = v(\xi)$, $\xi = x - ct$

$$u_t = -c v'(\xi), \quad u_x = v'(\xi)$$

$$-cv' + vv' = \gamma v''$$

Integrate: $-cv + \frac{1}{2}v^2 = \gamma v' + k$

$$\gamma v' = k - cv + \frac{1}{2}v^2$$

As $\xi \to \pm\infty$, require $v' \to 0$ (bounded). Hence,

$$v \to c \pm \sqrt{c^2 - 2k} = \begin{cases} b \\ a \end{cases} \quad \text{as } |\xi| \to \infty.$$

Thus we need $k \leq \frac{1}{2}c^2$. (Take $k < \frac{1}{2}c^2$ otherwise $v = \text{const.}$)

$$2\nu v' = 2h - 2cv + v^2 = (v-a)(v-b)$$

$$c = \frac{1}{2}(a+b), \qquad h = \frac{1}{2}ab, \qquad a < b$$

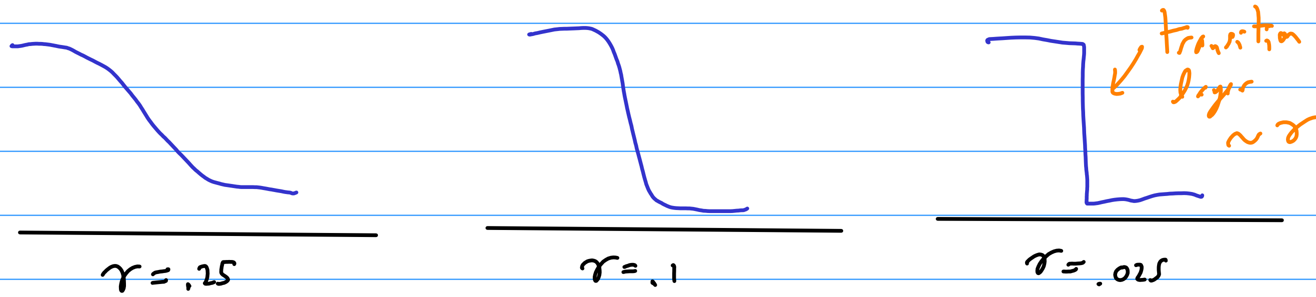
$$\int \frac{2\nu \, dv}{(v-a)(v-b)} = \frac{2\nu}{b-a} \log\left(\frac{b-v}{v-a}\right) = \xi - \delta$$

$$v(\xi) = \frac{a e^{(b-a)(\xi-\delta)/2\nu} + b}{e^{(b-a)(\xi-\delta)/2\nu} + 1} = u(t,x)$$

$$\xi = x - ct$$

$$\lim_{t \to -\infty} u(t,x) = b, \qquad \lim_{t \to \infty} u(t,x) = a$$

So $c = \frac{1}{2}(a+b)$ is the average of the two speeds at ∞.



Converge to step shock-wave solution as $\nu \to 0$!

(Conserves mass across jump, like Rankine–Hugoniot condition.)

Hopf-Cole transformation:

$$v_t = \gamma v_{xx}$$

Put $v(t,x) = e^{\alpha \varphi(t,x)}$, $\quad \varphi(t,x) = \frac{1}{\alpha} \log v(t,x)$

real if $v > 0$. Ok if $v(0,x) > 0$.

$$v_t = \alpha \varphi_t v, \quad v_x = \alpha \varphi_x v, \quad v_{xx} = \alpha\left(\varphi_{xx} + \alpha \varphi_x^2\right) v$$

Hence:

$$\varphi_t = \gamma \varphi_{xx} + \gamma \alpha \varphi_x^2$$ potential Burgers' eq'n

Differentiate: $\varphi_{tx} = \gamma \varphi_{xxx} + 2\gamma\alpha \varphi_x \varphi_{xx}$

Now set $u = \varphi_x$.

$$u_t = \gamma u_{xx} + 2\gamma\alpha u u_x$$

Set $\alpha = -\frac{1}{2\gamma}$: get Burger's equation!

So the Hopf-Cole transformation turns Burgers' (non-l) into the heat equation (linear)

Any $v(t,x) > 0$ which solves the heat equation $v_t = \gamma v_{xx}$ gives

$$u(t,x) = \frac{\partial}{\partial x}\left[-2\gamma \log v(t,x)\right] = -2\gamma \, v_x / v$$

which solves Burgers' eq'n $u_t + u u_x = \gamma u_{xx}$.

Do all solutions arise in this way?

Take $u(t,x)$ which solves Burgers', and let

$$\tilde{\varphi}(t,x) = \int_0^x u(t,x)\,dx, \qquad \tilde{\varphi}_x = u$$

$$\tilde{\varphi}_{xt} + \tilde{\varphi}_x \tilde{\varphi}_{xx} = \gamma \tilde{\varphi}_{xxx}$$

$$\tilde{\varphi}_t + \frac{1}{2}\tilde{\varphi}_x^2 = \gamma \tilde{\varphi}_{xx} + g(t)$$

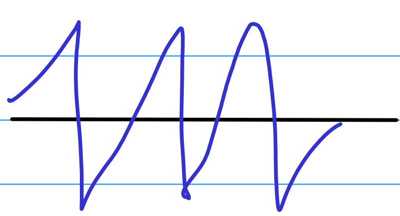
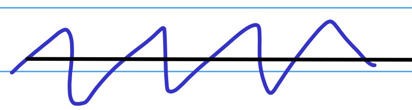
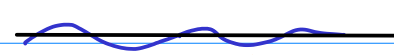
This is not the same as potential Burgers because of $g(t)$, but then just modify potential: $\varphi = \tilde{\varphi} - G(t)$, $G' = g$.

So the answer is yes!

example: $v(t,x) = a + b e^{-\gamma \omega^2 t} \cos \omega x$ $\qquad a > b$ so $v > 0$

$$\varphi = -2\gamma \log v$$

$$\varphi_x = -2\gamma \frac{v_x}{v} = \frac{2\gamma b \omega e^{-\gamma\omega^2 t} \sin \omega x}{a + b e^{-\gamma\omega^2 t}\cos \omega x}$$

$2\gamma\left(\frac{b}{a}\right)\omega e^{-\gamma\omega^2 t}\sin \omega x$

diffusive sol'n, since amplitude small

See book for explicit shock example