Lecture 23: Symmetry & similarity

$\hat{t} = t + a$
$\hat{x} = x + b$

Translation: $t \to t + a, \quad x \to x + b$

$$U(\hat{t}, \hat{x}) = u(\hat{t} - a, \hat{x} - b) \quad \text{then drop } \wedge$$

$$\frac{\partial U}{\partial t} = \frac{\partial u}{\partial t}, \quad \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$

Hence if u is a solution of $u_t = \gamma u_{xx}$, then
U " " " " $U_t = \gamma U_{xx}$.

$$u(t,x) = e^{-\gamma t} \sin x \quad \Rightarrow \quad U(t,x) = e^{-\gamma(t-a)} \sin(x - b)$$

→ infinite family of solutions

We say that the heat equation is symmetric (invariant) under time and space translation.

(Warning: BCs can break this.)

Scaling invariance: $U(t,x) = c\,u(t,x) + k$ ← linear equation

(change in temperature scale)

Not so interesting

More interesting: $t \to \alpha t$, $x \to \beta x$

$$U(t,x) = u(\alpha^{-1} t, \beta^{-1} x)$$

$$U_t = \frac{1}{\alpha} u_t, \quad U_x = \frac{1}{\beta} u_x, \quad U_{xx} = \frac{1}{\beta^2} u_{xx}$$

So $u_t = \gamma u_{xx} \Rightarrow \alpha U_t = \gamma \beta^2 U_{xx}$

$$U_t = \Gamma U_{xx}, \quad \Gamma = \gamma\beta^2/\alpha.$$

$\left.\begin{array}{l}\alpha = 2 \\ \beta = 1\end{array}\right\}$ diffuse twice as slow ($\Gamma = \gamma/2$)

$\left.\begin{array}{l}\alpha = 1 \\ \beta = 2\end{array}\right\}$ diffuse four times faster ($\Gamma = 4\beta$)

For $\alpha = \beta^2$, $\Gamma = \gamma$! So no change at all.

Heat equation invariant under rescaling $(t,x) \to (\beta^2 t, \beta x)$

"scaling symmetry"

(Note: $\beta \neq 1$ alters domain, so may violate BCs.)

Solutions that stay unchanged under rescaling are called similarity solutions.

If PDE admits scaling symmetry:

$$t \mapsto \beta^a t, \quad x \mapsto \beta^b x, \quad u \mapsto \beta^c u, \quad \beta \neq 0$$

a, b, c fixed, a, b not both 0, then if $u(t,x)$ is sol'n so is

$$U(t,x) = \beta^c u(\beta^{-a} t, \beta^{-b} x).$$

$$u_t \to \beta^{c-a} u_t, \quad u_x \to \beta^{c-b} u_x, \quad u_{tt} \to \beta^{c-2a} u_{tt}, \text{ etc.}$$

To admit symmetry, each term scales by same overall power of β.

Similarity solution: $u(t,x) = \beta^c u(\beta^{-a} t, \beta^{-b} x)$, $\forall \beta > 0$

$u(t,x)$ does not depend on β

Take $a \neq 0$, $b \neq 0$, $t > 0$

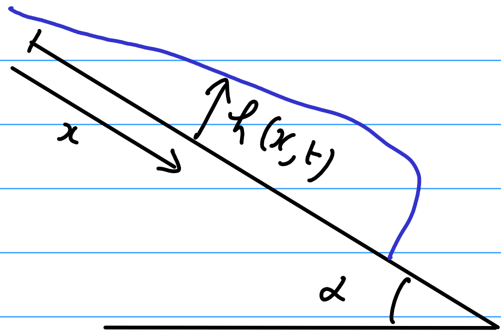
Take $\beta = t^{1/a}$: $u(t,x) = t^{c/a} u(1, t^{-b/a} x)$

$$= t^{c/a} v(\xi), \quad \xi = \frac{x}{t^{b/a}}$$

similarity variable

Now converting PDE to ξ coords makes it an ODE!

example: viscous flow down a slope (Acheson, p. 245)



"thin film"

start from Navier-Stokes

expand in thin h

Get approximate equation:

$$h_t + \frac{g \sin\alpha}{\nu} h^2 h_x = 0$$

↖ viscosity

Transport equation

Thicker regions propagate faster

Write $\gamma = \frac{g \sin\alpha}{\nu}$

Let $H(t,x) = \beta^{c} h(\beta^{-a} t, \beta^{-b} x)$

$$H_t = \beta^{c-a} h_t, \quad H_x = \beta^{c-b} h_x$$

$$\beta^{c-a} H_t + \gamma (\beta^{2c})(\beta^{c-b}) H_{xx} = 0$$

$$H_t + \gamma \beta^{2c-b+a} H_{xx} = 0$$

$2c - b + a = 0$ Take $a = b$: $c = 0$

$$h(t,x) = h(\beta^{-a} t, \beta^{-a} x)$$

$$= v(\xi), \quad \xi = x/t.$$

Plug back into equation:

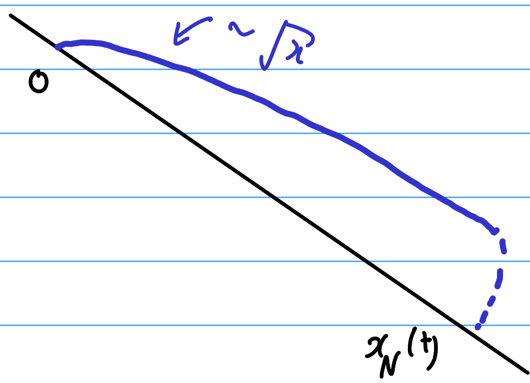
$$h_t + \gamma h^2 h_x = v'(\xi)\,\xi_t + \gamma v^2 v'(\xi)\,\xi_x$$

$$= v'(\xi)\left(\frac{-x}{t^2} + \gamma v^2(\xi)\,\frac{1}{t}\right) = 0$$

$v'(\xi) \neq 0$

$$\boxed{v^2(\xi) = \frac{1}{\gamma}\,\frac{x}{t}}$$

So $h = \left(\frac{\nu}{g \sin\alpha}\right)^{1/2} \sqrt{\frac{x}{t}}$ is a similarity solution. (non linear!)



This is actually the universal solution that any initial "blob" achieves, except for the nose.

$x_N(t)$ = "nose" position

<u>Spreading rate:</u>

$$\int_0^{x_N(t)} h\,dx = A$$

"mass" (area) conservation

$$x_N(t) = \left(\frac{9A^2 g \sin\alpha}{4\nu}\right)^{1/3} t^{1/3}$$