Lecture 21: Fundamental solutions

Fundamental solution is like Green's function for IVP:

$F(t,x;\xi)$: $\quad F_t = \gamma F_{xx}, \quad F(0,x;\xi) = \delta(x-\xi)$

Like releasing a concentrated pulse of heat at ξ, then watching it evolve.

Then IVP with $u(0,x) = f(x)$ has solution

$$u(t,x) = \int_a^b F(t,x;\xi)\, d\xi$$

Simple case: $\quad u_t = u_{xx}, \quad -\infty < x < \infty, \quad t > 0$

$u(0,x) = f(x) \qquad$ (require $\|u\| < \infty$)

The Fourier transform (FT) is

$$\hat{u}(t,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t,x) e^{-ikx}\, dx$$

convention!

(Limit $L \to \infty$ of periodic Fourier series)

Take FT of equation:

$$\hat{u}_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx} e^{-ikx} dx$$

$$= -k^2 \hat{u} \quad \text{after two integration by parts.}$$

Solve ODE: $\hat{u}(t,k) = \underbrace{\hat{u}(0,k)}_{\hat{f}(k)} e^{-k^2 t}$

Inverse transform:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(t,k) e^{ikx} dk$$

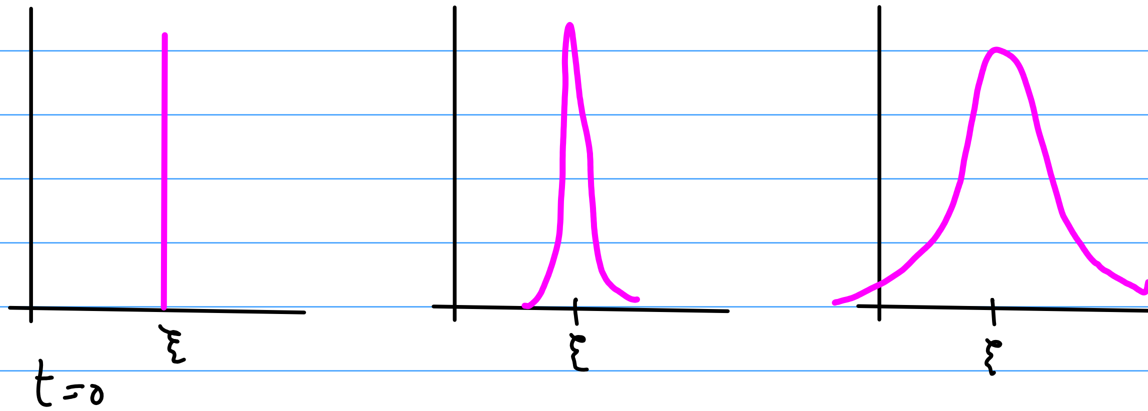
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx - k^2 t} dk$$

Now for $f(x) = \delta(x-\xi)$, we have $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-ik\xi}$

$$F(t,x;\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-\xi) - k^2 t} dk$$

$$\boxed{F(t,x;\xi) = \frac{1}{\sqrt{4\pi t}} e^{-(x-\xi)^2/4t}}$$

"heat kernel"



Heat spreads with a Gaussian profile.

Check directly: $\displaystyle\int_{-\infty}^{\infty} F(t,x;\xi)\,dx = 1$

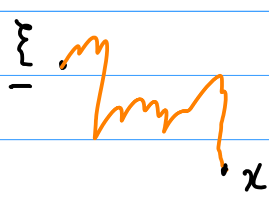
As $t \to 0$, width of Gaussian $\to 0$, so this is a representation of δ-function.

In dimension d,

$$F(t,\underline{x};\underline{\xi}) = \frac{1}{(4\pi t)^{d/2}}\, e^{-\|\underline{x}-\underline{\xi}\|^2/4t}$$

Probabilistic (Brownian motion) interpretation:

$$P(t,\underline{x};\underline{\xi}) = F(t,\underline{x};\underline{\xi})$$



$P\,d^d x$ is the probability of finding a "Brownian particle" at $\underline{x}$, given that it started at $\underline{\xi}$.

We can now write the general solution to IVP as

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-(x-\xi)^2/4t}\, d\xi$$

$$= f(x) * F_0(t;x) \qquad F_0(t;x) = F(t,x;0)$$

↑ convolution

Related problem: localized source

$$u_t = u_{xx} + \delta(x-\xi)\,\delta(t-\tau), \qquad u(0,x)=0, \quad \tau>0$$

Solution is general fundamental solution.

$$u(t,x) = G(t,x;\tau,\xi)$$

Note that $G(t,x;\tau,\xi)=0$, $t<\tau$.

Hence, for a general source $u_t = u_{xx} + h(t,x)$, $u(0,x)=f(x)$:

$$u(t,x) = \int_0^t \int_a^b G(t,x;\tau,\xi)\, h(\tau,\xi)\, d\xi\, d\tau$$

$$+ \int_a^b F(t,x;\xi)\, f(\xi)\, d\xi$$

← initial condition

Let's find G for the heat equation: first Fourier trans:

$$\hat{u}_t + k^2 \hat{u} = \frac{1}{\sqrt{2\pi}} e^{-ik\xi} \delta(t-\tau)$$

$$\hat{u}(0,k) = 0$$

$$\left(e^{k^2 t} \hat{u}\right)_t = \frac{1}{\sqrt{2\pi}} e^{-ik\xi + k^2 t} \delta(t-\tau)$$

$$e^{k^2 t} \hat{u} = \frac{1}{\sqrt{2\pi}} \int_0^t e^{-ik\xi + k^2 s} \delta(s-\tau)\, ds$$

(=0 for $t<\tau$)

$$= \frac{1}{\sqrt{2\pi}} e^{-ik\xi + k^2 \tau} \sigma(t-\tau)$$

$$\sigma(s) = \begin{cases} 0, & s<0 \\ 1, & s>0 \end{cases}$$

Now take inverse Fourier transform:

$$u(t,x) = G(t,x;\tau,\xi) = \frac{\sigma(t-\tau)}{2\pi} \int_{-\infty}^{\infty} e^{-ik\xi - k^2(t-\tau)} e^{ikx}\, dk$$

$$\boxed{G(t,x;\tau,\xi) = \sigma(t-\tau)\, F(t,x;\xi)}$$

Translation of fundamental solution!

"Duhamel's principle"