Lecture 20: Conformal mappings (cont'd)

tired of writing $-\frac{1}{2\pi}$

example: corner of angle γ.

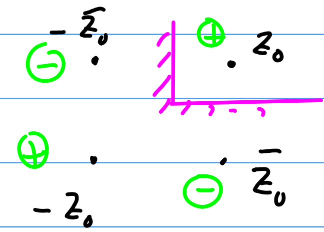
$$w = z^{\pi/\gamma}. \qquad \tilde{g}(w; w_0) = \log\left(\frac{w - w_0}{w - \bar{w}_0}\right)$$

$$g(z; z_0) = \log\left(\frac{z^{\pi/\gamma} - z_0^{\pi/\gamma}}{z^{\pi/\gamma} - \bar{z}_0^{\pi/\gamma}}\right)$$

Check: $\gamma = \pi/2$: $g = \log\left(\frac{z^2 - z_0^2}{z^2 - \bar{z}_0^2}\right)$

$$= \log\left(\frac{(z - z_0)(z + z_0)}{(z - \bar{z}_0)(z + \bar{z}_0)}\right)$$

So images at $-z_0$, $\pm \bar{z}_0$!



Same as before with method of images!

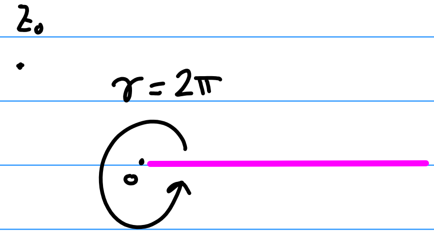
Except now this works for any $0 < \gamma < 2\pi$.

$\gamma = 2\pi$? Sounds like $\gamma = 0$ but let's try:

$$g(z; z_0) = \log\left(\frac{z^{1/2} - z_0^{1/2}}{z^{1/2} - \bar{z}_0^{1/2}}\right)$$

Now let $z = x > 0$:

$$\operatorname{Re} g(x; z_0) = \operatorname{Re} \log\left(\frac{x^{1/2} - z_0^{1/2}}{x^{1/2} - \bar{z}_0^{1/2}} \right) = 0$$ since have same length!



Thus the Green's function vanishes on the real axis, as if there was an infinitely thin conductor there!

Can think of this as Green's function for a thin "probe".

So where are the images? $z^{1/2} = \bar{z}_0^{1/2}$ in D? bad!

But remember square root: $w = z^{1/2}$

$$|w| e^{i\phi} = |z|^{1/2} e^{i\theta/2}$$

$$\phi = \frac{\theta}{2} + \pi m, \quad m \in \mathbb{Z}$$

"Normally": $\sqrt{e^{i\pi}} = e^{i\pi/2}$, $\sqrt{e^{-i\pi}} = e^{-i\pi/2}$

But notice that then $\arg \sqrt{z}$ "jumps" at $\theta = \pi$!

Need "branch cut" to enforce continuous square root.

branch cut along negative real axis
$\theta \in (-\pi, \pi)$.

For our "probe" we want continuity in the "physical" domain, so instead take

$$\theta \in (0, 2\pi)\,!$$

So now $z^{1/2} - \bar{z}_0^{1/2}$ in the denominator of Green's function is zero when:

$$|z|^{1/2} e^{i\theta/2} = |z_0|^{1/2} e^{-i\theta_0/2}$$

$$\underbrace{\frac{\theta}{2}}_{\in (0,\pi)} = \underbrace{-\frac{\theta_0}{2}}_{\in (-\pi, 0)}$$

The arguments are never equal! The image lives on a different "Riemann sheet".

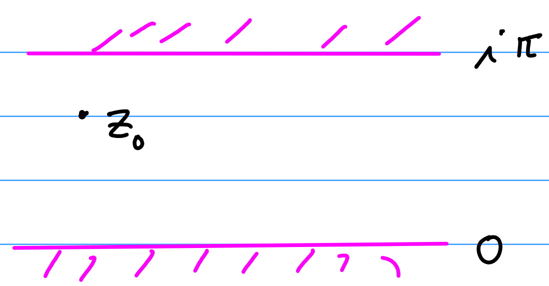
So for $z_0 = -1 = e^{i\pi}$, for example, denominator is

$$z^{1/2} - \bar{z}_0^{1/2} = z^{1/2} - e^{-i\pi/2} = z^{1/2} + i,$$

Not singular for $0 < \theta < 2\pi$!

One last example:

Point source between two plates.

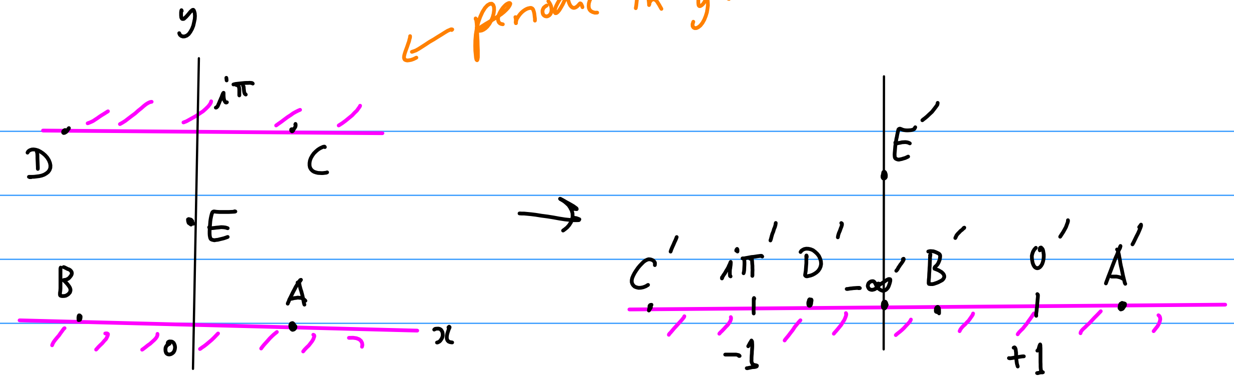


$$w = f(z) = e^{z}$$

$z = x$: $\quad w = e^{x} \in \mathbb{R},\ \geq 0$

$z = x + i\pi$: $\quad w = e^{x} e^{i\pi} = -e^{x} \in \mathbb{R},\ \leq 0$

← periodic in y!



Check interior: $z = i/2, \quad w = e^{i\pi/2} = i$

So Green's function is the real part of: $\left(-\frac{1}{2\pi} \times \dots\right)$

$$g(z; z_0) = \log\left(\frac{e^{z} - e^{z_0}}{e^{z} - e^{\bar{z}_0}}\right).$$

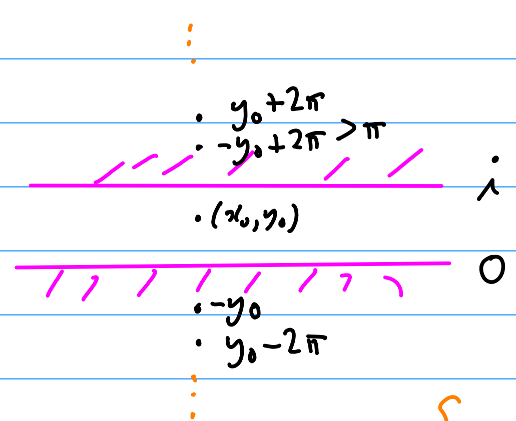
So where are the images? All singularities not in $\mathbb{R} \times [0, i]$.

$$e^{z} = e^{z_0} \iff e^{x} e^{iy} = e^{x_0} e^{iy_0} \qquad \begin{aligned} x &= x_0 \\ y &= \underbrace{y_0}_{(0,\pi)} + 2\pi i m, \quad m \neq 0 \end{aligned} \quad m \in \mathbb{Z}$$

double-reflection $0, \pi$

$$e^{z} = e^{\bar{z}_0} \iff e^{x} e^{iy} = e^{x_0} e^{-iy_0} \qquad \begin{aligned} x &= x_0 \\ y &= \underbrace{-y_0}_{(-\pi,0)} + 2\pi i m, \quad m \in \mathbb{Z} \end{aligned}$$

reflection about $0, \pm\pi, \pm 2\pi, \dots$



$y' = -y_0$ reflection about 0

$y'' = 2\pi - y' = 2\pi + y_0$ reflection about π

So $y'' = y_0 + 2\pi$ is a "double reflection"

$$g(z;z_0) = \log\left(\frac{e^{x}e^{iy} - e^{x_0}e^{iy_0}}{e^{x}e^{iy} - e^{x_0}e^{-iy_0}}\right)$$

$$\left|e^{x}e^{iy} - e^{x_0}e^{iy_0}\right|^2 = \left(e^{x}e^{iy} - e^{x_0}e^{iy_0}\right)\left(e^{x}e^{-iy} - e^{x_0}e^{-iy_0}\right)$$

$$= e^{2x} + e^{2x_0} - e^{x+x_0}e^{i(y-y_0)} - e^{x+x_0}e^{-i(y-y_0)}$$

$$= e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y-y_0)$$

Hence, the real Green's function is:

$$G(x,y;x_0,y_0) = \operatorname{Re} g = \log\left(\frac{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y-y_0)}{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y+y_0)}\right)$$

Large x:

$$G \sim \log\left(\frac{1 + e^{2(x_0-x)} - 2e^{x_0-x}\cos(y-y_0)}{1 + e^{2(x_0-x)} - 2e^{x_0-x}\cos(y+y_0)}\right)$$

Let $\varepsilon = e^{x_0-x}$. $\quad G \sim \log\left(\frac{1 + \cancel{\varepsilon^2} - 2\varepsilon\cos(y-y_0)}{1 + \cancel{\varepsilon^2} - 2\varepsilon\cos(y+y_0)}\right)$

neglect

$$G \sim -\left(\cos(y-y_0) - \cos(y+y_0)\right)\varepsilon + O(\varepsilon^2) = -2\sin y \sin y_0\, e^{x_0-x} + O(\varepsilon^2)$$

$x \gg x_0$

"Confinement" leads to exponential decay. $\left(\times -\frac{1}{2\pi}\right)$