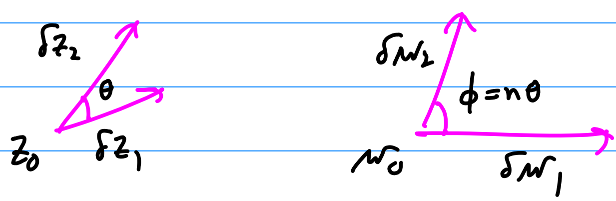
Lecture 19: Conformal mappings

$$w = f(z)$$

These preserve angles between vectors (say, tangent vectors), except when $f' = 0$.

When $f' = 0$, $\quad f(z) = f(z_0) + \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n + \ldots$



Angles increased by n.

$g(z_0) \neq 0$

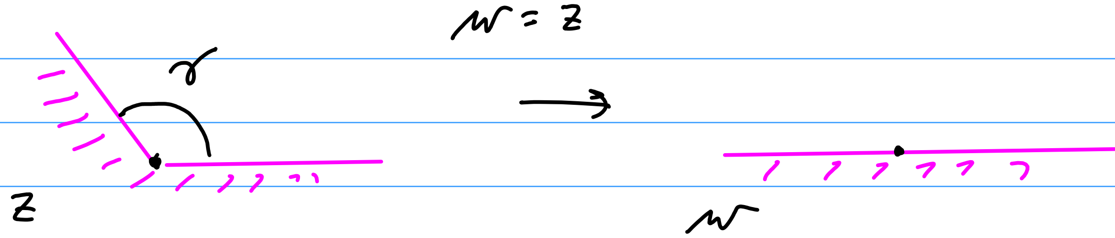
More generally, if $\quad f(z) = w_0 + (z - z_0)^{\alpha} g(z)$

$$z = z_0 + \delta z, \quad w = w_0 + \delta w$$

$$\cancel{w_0} + \delta w = \cancel{f(z_0)} + (\delta z)^{\alpha} \left(g(z_0) + O(\delta z) \right)$$

So $\arg \delta w = \alpha \arg \delta z$, i.e., angles multiplied by α.

$$w = z^{\pi/\gamma}$$



Note that $f(z)$ is locally invertible around a regular point, but not around a critical point:

example: $w = z^{\pi/\gamma}$, $z = 1$ and $z = e^{i\theta_m}$ map to 1,

with $\theta_m = 2m\gamma$, $m = 0, 1, \cdots, \left\lceil \frac{\pi}{\gamma} \right\rceil - 1$

That's ok! We keep to a single sector in the z-plane.

Riemann mapping theorem: D simply-connected domain in $\mathbb{C}$ (not all of $\mathbb{C}$). Then $\exists$ $w = f(z)$ that maps D to upper-half w-plane $\operatorname{Im} w > 0$.

Proof is not constructive!

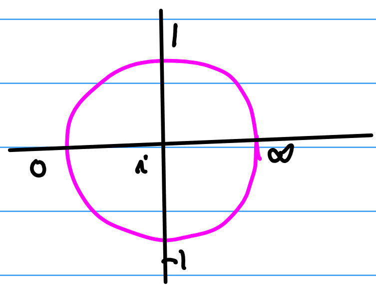
example: Take D = unit disk $|z| < 1$!

$$w = f(z) = i \frac{1+z}{1-z}.$$ Where is D mapped?

First note $f(0) = i$, so interior will map to upper-half plane if boundary maps to $\mathbb{R}$.

$$z = e^{i\theta}: \quad w = i\,\frac{1+e^{i\theta}}{1-e^{-i\theta}} = i\left(\frac{e^{i\theta/2}\left(e^{i\theta/2}+e^{-i\theta/2}\right)}{e^{i\theta/2}\left(e^{i\theta/2}-e^{-i\theta/2}\right)}\right)$$

$$w = i\,\frac{2\cos(\theta/2)}{2i\sin(\theta/2)} = \cot(\theta/2) \in \mathbb{R}\,!$$



So let's find the Green's function for a point-source in a disk.

First, $\tilde{g}(w; w_0) = -\frac{1}{2\pi}\log\left(\frac{w-w_0}{w-\bar{w}_0}\right)$ (real part)

on the upper-half w-plane. (image method)

$$\tilde{g}(f(z); f(z_0)) = -\frac{1}{2\pi}\log\left(\frac{i\left(\frac{1+z}{1-z}\right) - i\left(\frac{1+z_0}{1-z_0}\right)}{i\left(\frac{1+z}{1-z}\right) + i\left(\frac{1+\bar{z}_0}{1-\bar{z}_0}\right)}\right)$$

$$= -\frac{1}{2\pi}\log\left(\frac{\dfrac{(1+z)(1-z_0) - (1+z_0)(1-z)}{\cancel{(1-z)}(1-z_0)}}{\dfrac{(1+z)(1-\bar{z}_0) + (1+\bar{z}_0)(1-z)}{\cancel{(1-z)}(1-\bar{z}_0)}}\right)$$

$$g(z;z_0) = -\frac{1}{2\pi}\log\left(\left(\frac{z-z_0}{1-z_0}\right)\Big/\left(\frac{1-z\bar{z}_0}{1-\bar{z}_0}\right)\right)$$

$$\operatorname{Re} g(z;z_0) = -\frac{1}{2\pi}\log\frac{|z-z_0|}{|1-z\bar{z}_0|}$$

<u>Check:</u> $z = e^{i\theta} \Rightarrow \frac{1}{z} = \bar{z}$. $\operatorname{Re} g(e^{i\theta};z_0) = -\frac{1}{2\pi}\log\frac{|z-z_0|}{|z|\,|\bar{z}-\bar{z}_0|} = 0.$