Lecture 18: Complex variable methods

2D harmonic problems are the perfect setting for complex variables.

$$\phi(z) = u(x,y) + i\,v(x,y)$$

If $\phi(z)$ is analytic, its real and imaginary parts satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This comes from defining $\phi'(z)$ unambiguously:

$$\phi'(z) = \lim_{\Delta z \to 0} \frac{\phi(z+\Delta z) - \phi(z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}.$$

But also, by letting $\Delta z = i\Delta y$,

$$\phi'(z) = \lim_{\Delta y \to 0} \frac{\phi(z+i\Delta y) - \phi(z)}{i\Delta y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$

Equating real/imag. parts gives the C-R equations.

It follows immediately that

$$\Delta u = u_{xx} + u_{yy} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial v}{\partial x}\right) = 0$$

$$\Delta v = v_{xx} + v_{yy} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0$$

(As long as derivatives exist)

Hence, both the real and imaginary parts of $f(z)$ are harmonic!

example:

$G(\underline{x};\underline{\xi}) = -\frac{1}{2\pi}\log\|\underline{x}-\underline{\xi}\|$ from last time

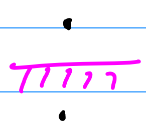
can be written as the real part of

$$g(z;z_0) = -\frac{1}{2\pi}\log(z-z_0).$$

$$= -\frac{1}{2\pi}\left(\log|z-z_0| + i\arg(z-z_0)\right)$$

$\arctan\left(\frac{y-y_0}{x-x_0}\right)$

Image method: $\Omega = \{y>0\}$



$$g = -\frac{1}{2\pi}\left(\log(z-z_0) - \log(z-\bar{z}_0)\right)$$

example:

$\phi(z) = z + z^{-1}$

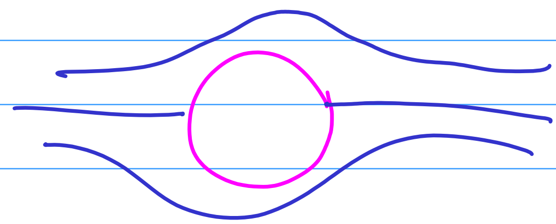
Unit circle: $\phi(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\sin\theta \in \mathbb{R}$!

This means $\operatorname{Im}\phi(z) = 0$ on unit circle C.

Hence, $\operatorname{Im}\phi(z)$ satisfies Dirichlet problem for $|z|>1$.

$$\operatorname{Im}\phi(z) = \operatorname{Im}\left(re^{i\theta} + r^{-1}e^{-i\theta}\right) = \left(r - r^{-1}\right)\sin\theta$$

This is actually the electric potential for a field with a an insulating obstacle:



(or flow around a cylinder)

What about the real part? $\phi = u + iv$, $u = \operatorname{Re}\phi$

$$\underline{r}\cdot\nabla u = x u_x + y u_y = \operatorname{Re}\left[(x+iy)(u_x - iu_y)\right]$$
$$= \operatorname{Re}\left[z\,(u_x + iv_x)\right] = \operatorname{Re}\left[z\,\phi_x\right] = \operatorname{Re}\left[z\phi'(z)\right]$$

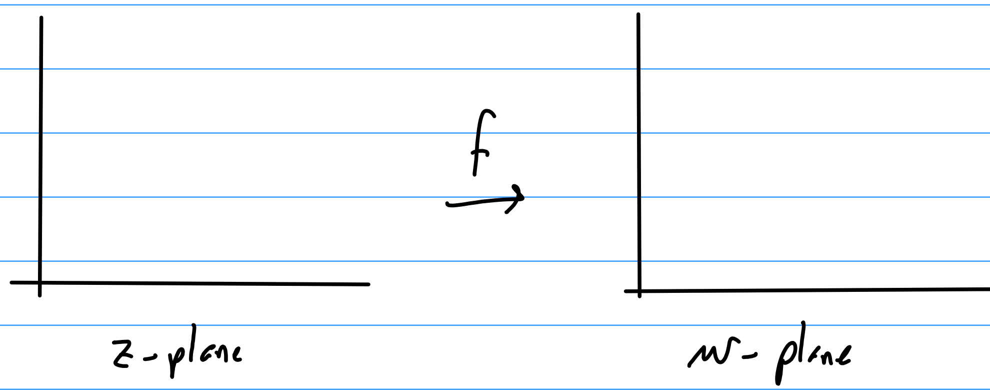
For $\phi(z) = z + z^{-1}$, $\phi'(z) = 1 - z^{-2}$, so $z\phi'(z) = z - z^{-1}$.

$$\underline{r}\cdot\nabla u\Big|_C = \operatorname{Re}\left[z\phi'(z)\Big|_{z=e^{i\theta}}\right] = \operatorname{Re}\left[\underbrace{e^{i\theta} - e^{-i\theta}}_{2i\sin\theta}\right] = 0!$$

Conclude: $\operatorname{Re}\phi$ satisfies Neumann B.C.!

How to show this in general? Is it true?

Conformal transformations:



$w = f(z)$ is a conformal transformation.

Note that $\phi(f(z))$ is analytic if $\phi(w)$ is, since both functions of a single complex variable!

Hence, conformal transformations map analytic functions to other analytic functions.

Consider a point $z = z_0 + \delta z$.

$$w = w_0 + \delta w = f(z_0 + \delta z) = f(z_0) + f'(z_0)\,\delta z + \cdots$$

Since $w_0 = f(z_0)$, we have $\delta w = f'(z_0)\,\delta z$.

Now let $\delta z = |\delta z|\, e^{i\theta}$, $\delta w = |\delta w|\, e^{i\phi}$.

$$|\delta w|\, e^{i\phi} = f'(z_0)\, |\delta z|\, e^{i\theta}$$

$$\log|\delta w| + i\phi = \log|\delta z| + \log f'(z_0) + i\theta$$

Hence, $\phi = \theta + \underbrace{\mathrm{Im}\left(\log f'(z_0)\right)}_{\arg f'(z_0)}$

$$\boxed{\phi = \theta + \arg f'(z_0)}$$

C.T. "rotates" vectors at z_0 by uniform angle

It is said that CTs "preserve angles" between vectors

BUT: what if $f'(z_0) = 0$? Then arg not defined

<u>CRITICAL POINTS</u>. Assume $f^{(m)}(z_0) = 0$, $0 < m < n$.

$$\cancel{w_0} + \delta w = \cancel{f(z_0)} + \frac{1}{n!} f^{(n)}(z_0)\,(\delta z)^n + \cdots$$

$$|\delta w|\, e^{i\phi} = \frac{1}{n!} f^{(n)}(z_0)\,|\delta z|^n\, e^{in\theta} + \cdots$$

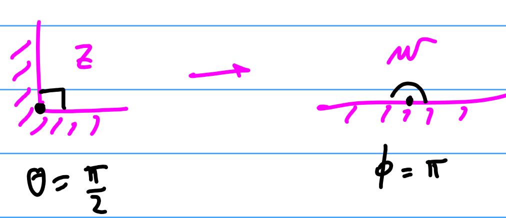
$$\log|\delta w| + i\phi = -\log(n!) + n\log|\delta z| + \log f^{(n)}(z_0) + n\theta$$

$$\boxed{\phi = n\theta + \arg f^{(n)}(z_0)}$$

Now the angle between vectors changes, but only at critical points!

<u>example:</u> $w = z^2$

$f'(0) = 0$



$n = 2$