Lecture 17: 2D Green's functions

$$-\Delta u = \delta(\underline{x}-\underline{\xi}) \quad \text{Poisson}$$

or $\quad -(u_{xx}+u_{yy}) = \delta(x-\xi)\,\delta(y-\eta)$

Write $\quad u = G(x,y;\xi,\eta) = G(\underline{x},\underline{\xi})$

Then solve $\quad -\Delta u = f \quad$ by

$$u(\underline{x}) = \int_\Omega G(\underline{x};\underline{\xi})\, f(\underline{\xi})\, d\xi\, d\eta$$

"Free-space": take $\Omega = \mathbb{R}^2$, $\underline{\xi} = \underline{0}$.

not relevant

By symmetry: $\quad G_0(\underline{x}) = v(r)$

$$\Delta v = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = 0 \quad \Rightarrow \quad v(r) = a + b\log r$$

To determine b such that $-\Delta v = \delta_0$, use

$$1 = \int_{D_\varepsilon} \delta_0(\underline{x})\, dx\, dy = -\int_{D_\varepsilon} \Delta v\, dx\, dy = -\int_{\partial D_\varepsilon} \nabla v \cdot \underline{n}\, ds$$

disk of radius ε

$$\underline{n}\cdot\nabla v = v'(r), \quad ds = \varepsilon\, d\theta$$

$$1 = -\int_{-\pi}^{\pi} N'(\varepsilon)\,\varepsilon\,d\theta = -2\pi\,\varepsilon N'(\varepsilon) = -2\pi b$$

$$\Rightarrow \quad b = -1/2\pi$$

Hence,
$$G(\underline{x};\underline{0}) = -\frac{1}{2\pi}\log\|\underline{x}\|$$

$$\boxed{G(\underline{x};\underline{\xi}) = -\frac{1}{2\pi}\log\|\underline{x}-\underline{\xi}\|}$$

This is, for example, the gravitational potential due to a point mass, or electric potential due to point charge, or heat due to point source.

$$\text{Force} = \nabla G = -\frac{1}{2\pi}\frac{\underline{x}-\underline{\xi}}{\|\underline{x}-\underline{\xi}\|^2} \sim \frac{1}{r}$$

What about a bounded region $\Omega \subset \mathbb{R}^2$?

Solve $\quad -\Delta G_{\underline{\xi}} = \delta(\underline{x}-\underline{\xi}), \qquad G_{\underline{\xi}}(\underline{x}) = 0$ on $\partial\Omega$.

Let $G_0(\underline{x};\underline{\xi}) = -\frac{1}{2\pi}\log\|\underline{x}-\underline{\xi}\|$.

Write $G(\underline{x};\underline{\xi}) = G_0(\underline{x};\underline{\xi}) + z(\underline{x};\underline{\xi})$.

Then $-\Delta G = -\Delta G_0 + \Delta z$

$$= \delta(\underline{x} - \underline{\xi}) + \Delta z$$

So choose $\Delta z = 0$ (harmonic function) in Ω.

Now z must be chosen such that $G_0 + z\big|_{\partial\Omega} = 0$,

but z is completely regular (no δ-function in Δz).

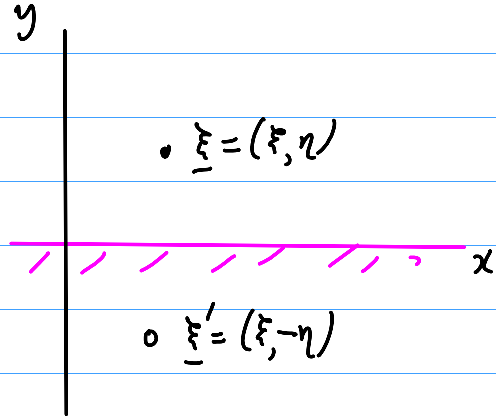
But how to find $z_{\underline{\xi}}(\underline{x})$?

Sometimes can use <u>method of images</u>

<u>example:</u> solve $-\Delta G_{\underline{\xi}} = \delta(\underline{x} - \underline{\xi})$ in $\Omega = \{(x,y) \mid y > 0\}$

with $G_{\underline{\xi}} = 0$ on $\partial\Omega$.

Write $G_{\underline{\xi}} = G_{0\underline{\xi}} + z_{\underline{\xi}}$. Argue by symmetry that to satisfy B.C. we can place a "source" at $\underline{\xi}' = (\xi, -\eta)$.



$$z_{\underline{\xi}} = -G_{0\underline{\xi}'}$$

Note that $\Delta z_{\underline{\xi}} = -\Delta G_{0_{\underline{\xi}'}} = \delta(\underline{x} - \underline{\xi}') = 0$ in Ω

Hence:

$$G_{\underline{\xi}}(\underline{x}) = -\frac{1}{2\pi} \log\left(\frac{\|\underline{x} - \underline{\xi}\|}{\|\underline{x} - \underline{\xi}'\|}\right), \qquad \underline{\xi}' = (\xi, -\eta)$$

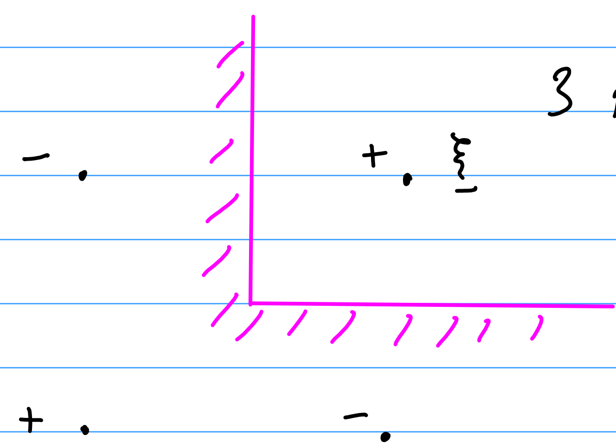
Check:

$$G_{\underline{\xi}}(x, 0) = -\frac{1}{4\pi} \log\left|(x-\xi)^2 + \eta^2\right| + \frac{1}{4\pi} \log\left|(x-\xi)^2 + (-\eta)^2\right|$$

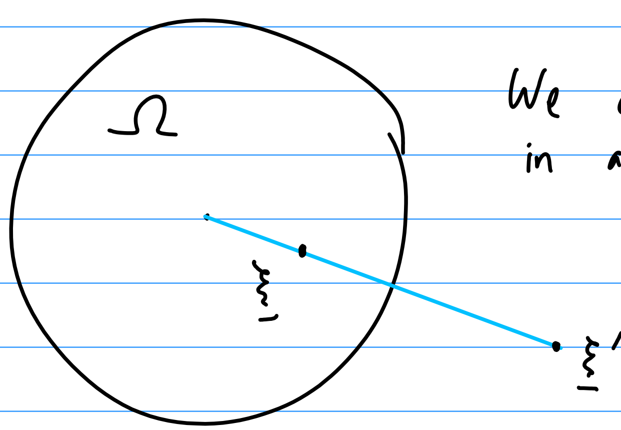
$$= 0.$$

For Neumann conditions, use $G_{\underline{\xi}}(\underline{x}) = G_{0_{\underline{\xi}}}(\underline{x}) + G_{0_{\underline{\xi}'}}(\underline{x})$.

Quadrant:



3 images are needed!



We can also use images in a disk. Try with one image.

Let $\underline{\xi}' = c\,\underline{\xi}$, by symmetry.

$$z = -a\log\|\underline{x}-\underline{\xi}'\| + \log b$$

$$G_0 + z = \log\|\underline{x}-\underline{\xi}\| - a\log\|\underline{x}-\underline{\xi}'\| - \log b$$

$$= \log\left(\frac{1}{b}\frac{\|\underline{x}-\underline{\xi}\|}{\|\underline{x}-\underline{\xi}'\|^a}\right)$$ (drop $\frac{-1}{2\pi}$ for now)

$c > 1$

On $\partial\Omega$: $\underline{x} = (\cos\theta, \sin\theta)$. Take $\underline{\xi} = (\xi, 0)$, $\underline{\xi}' = (c\xi, 0)$.

$$\|\underline{x}-\underline{\xi}\|^2 = (\cos\theta - \xi)^2 + \sin^2\theta = 1 + \xi^2 - 2\xi\cos\theta, \quad \|\underline{x}-\underline{\xi}'\|^2 = (\cos\theta - c\xi)^2 + \sin^2\theta = 1 + c^2\xi^2 - 2c\xi\cos\theta$$

Try with $a = 1$:

$$1 + \xi^2 - 2\xi\cos\theta = b^2(1 + c^2\xi^2 - 2c\xi\cos\theta)$$

$b^2 c = 1$: $\quad 1 + \xi^2 = b^2 + \frac{1}{b^2}\xi^2 \Rightarrow b = \xi$!

So for $\underline{\xi} = (\xi, 0)$, $\underline{\xi}' = \left(\frac{1}{\xi}, 0\right) = \frac{1}{|\xi|^2}(\xi, 0)$

$$\underline{\xi} = (\xi, \eta), \quad \underline{\xi}' = \frac{1}{\|\underline{\xi}\|^2}\underline{\xi}$$

$$G_{\underline{\xi}}(\underline{x}) = -\frac{1}{2\pi}\log\left(\frac{\|\underline{x}-\underline{\xi}\|}{\|\underline{\xi}\|\,\|\underline{x}-\underline{\xi}/\|\underline{\xi}\|^2\|}\right)$$

$$\boxed{G_{\underline{\xi}}(\underline{x}) = -\frac{1}{2\pi}\log\left(\frac{\|\underline{\xi}\|\,\|\underline{x}-\underline{\xi}\|}{\|\,\|\underline{\xi}\|^2\underline{x}-\underline{\xi}\,\|}\right)}$$