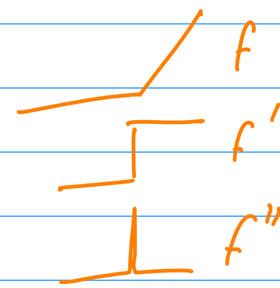
Lecture 16: Green('s) functions

<u>Fun:</u> "The Green of Green functions", Physics Today <u>56</u>, 12, 41 (2003)

<u>Amazing book:</u> "Green's functions and BVPs", Stakgold & Holst (850p.!)

Method for solving linear inhomogeneous problems



Start with example:

$$-u'' + \omega^2 u = f(x), \qquad u(0) = u(1) = 0$$

First solve problem with δ-function source:

$$-u'' + \omega^2 u = \delta(x-\xi), \qquad u(0) = u(1) = 0$$
$$0 < \xi < 1$$

So whenever $x \neq \xi$, just have homogeneous eq'n.

$$u(x) = A \cosh \omega x + B \sinh \omega x$$

However, use different solutions for $x < \xi$, $x > \xi$, then join them up to create δ-function.

Take $u(x) = G(x;\xi) = \begin{cases} a \sinh \omega x, & x \leq \xi \\ b \sinh \omega(1-x), & x \geq \xi \end{cases}$

Now u must be continuous: if it's discontinuous then u'' is δ'! (Too singular)

So demand continuity at $x = \xi$:

$$\text{1} \quad a \sinh \omega\xi = b \sinh \omega(1-\xi)$$

Is u' continuous? Examine ODE by integrating:

$$\underbrace{-\int_{\xi-\varepsilon}^{\xi+\varepsilon} u''\,dx}_{\text{"jump" in } u'} \; \underbrace{+\,\omega^2\int_{\xi-\varepsilon}^{\xi+\varepsilon} u\,dx}_{\to 0 \text{ as } \varepsilon\to 0 \text{ (continuous)}} = \underbrace{\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\,dx}_{1}$$

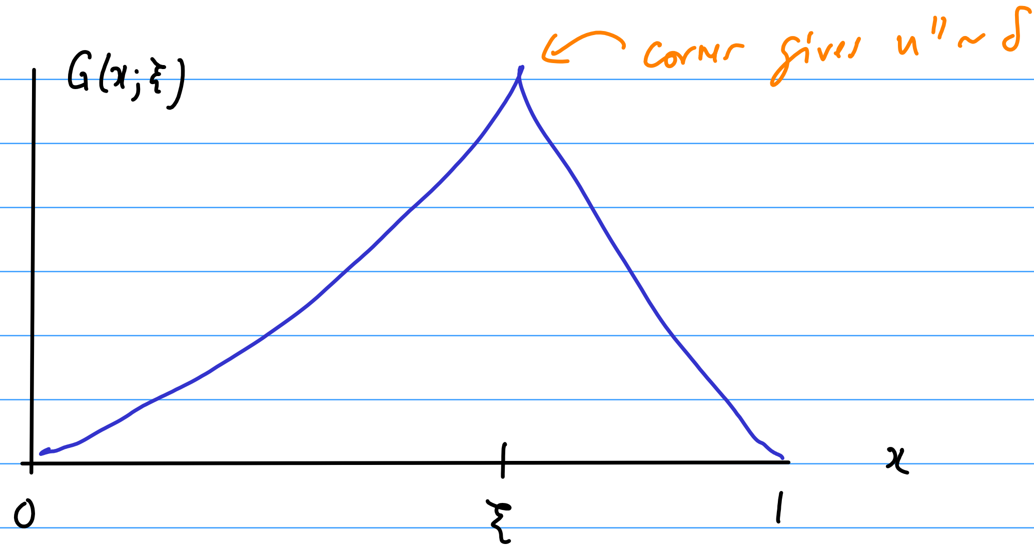
$$-\left(u'(\xi^+) - u'(\xi^-)\right) = 1$$

$$\text{2} \quad +\,\omega b \cosh \omega(1-\xi) + \omega a \cosh \xi = 1$$

Now there is some algebra to solve 1 and 2, using some hyperbolic trig ids. like $\sinh(\alpha+\beta) = \mathrm{sh}\,\alpha\,\mathrm{ch}\,\beta + \mathrm{sh}\,\beta\,\mathrm{ch}\,\alpha$.

$$G(x;\xi) = \begin{cases} \dfrac{\sinh \omega x \,\sinh \omega(1-\xi)}{\omega \sinh \omega}, & x \le \xi \\[2ex] \dfrac{\sinh \omega(1-x)\,\sinh \omega\xi}{\omega \sinh \omega}, & x \ge \xi \end{cases}$$

Note $G(x;\xi) = G(\xi;x)$ not accidental. (Self-adjoint problem)



The solution is like a "point source".

Now to solve $-u'' + \omega^2 u = f(x)$ we use

$$u(x) = \int_0^1 f(\xi)\, G(x;\xi)\, d\xi$$

$G_\xi(x) = G(x;\xi)$

Check: $$-u'' + \omega^2 u = \int_0^1 f(\xi) \underbrace{\left(-G_\xi'' + \omega^2 G_\xi\right)}_{\delta(x-\xi)} d\xi = f(x)$$

Of course, the resulting integral can be difficult, but at least can be manipulated.

$f(x) = 1 \Rightarrow$

$$u(x) = \int_0^x + \int_x^1 = \cdots = \frac{1}{\omega^2} - \frac{\sinh \omega x + \sinh \omega(1-x)}{\omega^2 \sinh \omega}$$

A Green's function approach is not always possible:

$$-cu'' = \delta(x-\xi), \qquad u'(0) = u'(1) = 0.$$

$$u(x) = -\frac{\rho(x-\xi)}{c} + ax + b \qquad \rho(x) = \begin{cases} 0, & x<0 \\ x, & x \geq 0 \end{cases}$$

ramp function

$$u'(0) = a = 0, \quad u'(1) = \underbrace{-\frac{1}{c}}_{\text{bad!}} + a = 0$$

No solution!

The problem is that in this Neumann case the homogeneous solution is non unique. Indeed, general solution of $-cu'' = f$, $u'(0)=u'(1)=0$,

$$u(x) = ax + b - \frac{1}{c}\int_0^x \int_0^y f(z)\,dz\,dy$$

$$u'(x) = a - \frac{1}{c}\int_0^x f(z)\,dz$$

$$u'(0) = 0 = a, \qquad u'(1) = 0 = -\frac{1}{c}\int_0^1 f(z)\,dz.$$

Hence, require $\int_0^1 f(z)\,dz = 0$ Example of "Fredholm alternative"

Solvability condition

If this holds, then solution is not unique! (b unconstrained)

Physically, some kind of mechanical, flux, etc. balance is needed.

The problem, then, is that $-cu'' = f$ with $f = \delta$ does not satisfy the solvability condition $\int f = 0$.

Enforce by solving instead: $-cu'' = \delta(x-\xi) - 1$, $u'(0) = u'(1) = 0$.

Away from ξ: $-cu'' = -1 \Rightarrow -cu = -\frac{1}{2}x^2 + \tilde{A} + \tilde{B}x$

$$G(x;\xi) = \begin{cases} \frac{1}{2c}x^2 + A + Bx, & x < \xi \\ \frac{1}{2c}x^2 + C + Dx, & x > \xi \end{cases}$$

$G_\xi'(0) = B = 0$, $\quad G_\xi'(1) = \frac{1}{c} + D = 0 \Rightarrow D = -\frac{1}{c}$

Continuity: $A = C - \xi/c$

$$G(x;\xi) = \begin{cases} C - \frac{1}{c}\xi + \frac{1}{2c}x^2, & x < \xi \\ C - \frac{1}{c}x + \frac{1}{2c}x^2, & x > \xi \end{cases}$$

Jump condition automagically satisfied! Why?

So C is still free. Require $\int_0^1 G(x;\xi)\,dx = 0$, $\forall\, 0 < \xi < 1$

to make $G(x;\xi) = G(\xi;x)$ Not required!

$$G(x;\xi) = \begin{cases} \frac{1}{3c} - \frac{1}{c}\xi + \frac{1}{2c}(x^2 + \xi^2), & x < \xi \\ \frac{1}{3c} - \frac{1}{c}x + \frac{1}{2c}(x^2 + \xi^2), & x > \xi \end{cases}$$

Modified Green's function

Solution as before: $u = \int_0^1 G(x;\xi)\, f(\xi)\, d\xi$

$$-cu'' = \int_0^1 -c\,G_\xi''(x)\, f(\xi)\, d\xi = \int_0^1 (\delta(x-\xi) - 1)\, f(\xi)\, d\xi$$

$$= f(x) - 0$$ since $\int_0^1 f\, d\xi = 0$